

EXOTIC ELLIPTIC ALGEBRAS OF DIMENSION 4

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WITH AN APPENDIX BY DEREK TOMLIN

ABSTRACT. Let E be an elliptic curve defined over an algebraically closed field k whose characteristic is not 2 or 3. Let τ be a translation automorphism of E that is not of order 2. In a previous paper we studied an algebra $A = A(E, \tau)$ that depends on this data: $A(E, \tau) = (S(E, \tau) \otimes M_2(k))^\Gamma$ where $S(E, \tau)$ is the 4-dimensional Sklyanin algebra associated to (E, τ) , $M_2(k)$ is the ring of 2×2 matrices over k , and Γ is $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$ acting in a particular way as automorphisms of S and $M_2(k)$. The action of Γ on S is compatible with the translation action of the 2-torsion subgroup $E[2]$ on E . Following the ideas and results in papers of Artin-Tate-Van den Bergh, Smith-Stafford, and Levasseur-Smith, this paper examines the line modules, point modules, and fat point modules, over A , and their incidence relations. The right context for the results is non-commutative algebraic geometry: we view A as a homogeneous coordinate ring of a non-commutative analogue of \mathbb{P}^3 that we denote by $\text{Proj}_{nc}(A)$. Point modules and fat point modules determine “points” in $\text{Proj}_{nc}(A)$. Line modules determine “lines” in $\text{Proj}_{nc}(A)$. Line modules for A are in bijection with certain lines in $\mathbb{P}(A_1^*) \cong \mathbb{P}^3$ and therefore correspond to the closed points of a certain subscheme \mathbb{L} of the Grassmannian $\mathbb{G}(1, 3)$. Shelton-Vancliff call \mathbb{L} the line scheme for A . We show that \mathbb{L} is the union of 7 reduced and irreducible components, 3 quartic elliptic space curves and 4 plane conics in the ambient Plücker \mathbb{P}^5 , and that $\deg(\mathbb{L}) = 20$. The union of the lines corresponding to the points on each elliptic curve is an elliptic scroll in $\mathbb{P}(A_1^*)$. Thus, the lines on that elliptic scroll are in natural bijection with a corresponding family of line modules for A .

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1. INTRODUCTION

This paper is a continuation of [9].

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1.1. The 3- and 4-dimensional Sklyanin algebras are certain non-commutative graded algebras $S = \mathbb{C} + S_1 + S_2 + \cdots$ that have many of the same ring-theoretic and homological properties as the polynomial rings in 3 and 4 variables. They are among the most interesting algebras that have appeared in non-commutative algebraic geometry; their study has stimulated and justified many of the developments in that subject. Such a Sklyanin algebra determines, and is determined by, an elliptic curve, E , a translation automorphism, $\tau : E \rightarrow E$, and an invertible \mathcal{O}_E -module \mathcal{L} of degree 3 or 4, respectively. We will denote the algebra by $S(E, \tau, \mathcal{L})$.¹

The 3-dimensional Sklyanin algebras are, arguably, the most interesting 3-dimensional Artin-Schelter regular algebras [2]. In [3] and [4], Artin-Tate-Van den Bergh completed the classification of the 3-dimensional Artin-Schelter algebras having Hilbert series $(1-t)^{-3}$. Such algebras are now viewed as homogeneous coordinate rings of non-commutative deformations of the projective plane \mathbb{P}^2 . If S is such an algebra we write $\text{Proj}_{nc}(S)$ for the corresponding non-commutative deformation of \mathbb{P}^2 . Van den Bergh [45] showed that as S ranges over all 3-dimensional Artin-Schelter algebras, the spaces $\text{Proj}_{nc}(S)$ give *all* non-commutative deformations of \mathbb{P}^2 . The precise statement of Van den Bergh's Theorem involves the deformation theory of the category of $\text{Qcoh}(\mathbb{P}^2)$ of quasi-coherent sheaves on \mathbb{P}^2 ; the deformation theory of abelian categories was formulated and developed by Lowen and Van den Bergh in [23] and [24]. In [45], Van den Bergh also classifies the non-commutative deformations of $\mathbb{P}^1 \times \mathbb{P}^1$.

Moving beyond surfaces, it is natural to try to classify 4-dimensional Artin-Schelter regular algebras and non-commutative deformations of \mathbb{P}^3 . A bewildering variety of such algebras is known. A classification remains a long way off. The 4-dimensional Sklyanin algebras are, again, the most interesting such algebras (so far) and they have a very rich structure. The representation theory of $S(E, \tau, \mathcal{L})$ and, what is almost the same thing, the geometric structure of $\text{Proj}_{nc}(S(E, \tau, \mathcal{L}))$, is governed by the geometry of E and τ when E is embedded as a quartic curve in $\mathbb{P}(H^0(E, \mathcal{L})^*)$.

We refer the reader to [1] and [32] for overviews of the 3- and 4-dimensional Sklyanin algebras and the spaces $\text{Proj}_{nc}(S(E, \tau, \mathcal{L}))$.

1.2. The significance of the 4-dimensional Sklyanin algebras is amplified by the fact that they arise naturally in mathematical physics: Sklyanin discovered them in the context of quantum statistical mechanics and Baxter's elliptic solutions to the Yang-Baxter equation ([30], [31]). He also observed that “degenerate” versions of the 4-dimensional Sklyanin algebras lead to the quantized enveloping algebras $U_q(\mathfrak{sl}_2)$. A detailed examination of this degeneration procedure is carried out in [11].

Real forms of 4-dimensional Sklyanin algebras arise from a natural problem in non-commutative differential geometry. In [12], [13], and [14], Connes and Dubois-Violette define a non-commutative 3-sphere in terms of the cyclic-homology Chern character $\text{ch}_* : K_*(\mathcal{A}) \rightarrow \text{HC}(\mathcal{A})$. They show that non-commutative 3-spheres are “unit spheres” in an ambient non-commutative analogue of \mathbb{R}^4 that is defined in terms of a C^* -algebra whose defining relations are necessarily those of a 4-dimensional Sklyanin algebra. The finer structure of non-commutative 3-spheres is very rich and involves, as might be expected, elliptic curves, theta functions, and other algebro-geometric data. The results of Connes and Dubois-Violette in these three papers provide one of the few interactions between non-commutative algebraic geometry and non-commutative differential geometry. We expect further exploration of this interaction will prove fruitful.

1.3. Notation and conventions. In §§1.1 and 1.2, all algebraic objects were assumed to be vector spaces over the field of complex numbers. From now on we will work over an algebraically closed field k of characteristic $\neq 2, 3$. Always, E will denote an elliptic curve defined over k , $\tau : E \rightarrow E$ will denote a translation automorphism, $S(E, \tau)$ will denote the 4-dimensional Sklyanin algebra associated to (E, τ) , Γ will denote the group $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$, and $M_2(k)$ will denote the ring of 2×2

¹The isomorphism class of $S(E, \tau, \mathcal{L})$ depends only on the degree of \mathcal{L} . Since the case $\deg(\mathcal{L}) = 3$ will only appear in §1.1, and only the case $\deg(\mathcal{L}) = 4$ appears from §1.3 onward we will simply write $S(E, \tau)$ from §1.3 onward.

matrices over k . The group Γ acts in a particular way as automorphisms of $S(E, \tau)$ and $M_2(k)$ (see §§2.6 and 3.1).

As explained in [34], there is a natural copy of E embedded as a quartic curve in $\mathbb{P}(S_1^*) \cong \mathbb{P}^3$. We fix an identity o and a group law on E such that four points on E are coplanar if and only if their sum is o . We will identify τ with $\tau(o)$. We write $E[2]$ and $E[4]$ for the 2-torsion and 4-torsion subgroups of E . The action of Γ as automorphisms of S induces an action of Γ on $\mathbb{P}(S_1^*)$; that action sends E to itself, and its action on E is the same as the translation action of $E[2]$. We often identify Γ and $E[2]$.

1.4. The algebras in the title of this paper are the invariant subalgebras

$$A(E, \tau) := (S(E, \tau) \otimes M_2(k))^\Gamma.$$

A similar operation can be applied to $A(E, \tau)$. By [9, Prop. 6.2], $(A(E, \tau) \otimes M_2(k))^\Gamma \cong S(E, \tau)$.

These algebras were studied in some detail in the authors' previous paper [9], where they were denoted $\tilde{Q}(E, \tau)$, and in the Ph.D. thesis and subsequent paper of Andrew Davies, [16] and [15]. Those papers show that $A(E, \tau)$ has the same excellent homological properties that $S(E, \tau)$ has. For example, $A(E, \tau)$ and $S(E, \tau)$ are, like the polynomial ring in 4 variables, noetherian domains with Hilbert series $(1 - t)^{-4}$, Koszul algebras of global homological dimension 4, and so on.

1.5. In this paper we view $A(E, \tau)$ as a homogeneous coordinate ring of a non-commutative analogue of \mathbb{P}^3 that we denote by $\text{Proj}_{nc}(A(E, \tau))$, and examine the geometric features of $\text{Proj}_{nc}(A(E, \tau))$, particularly the ways in which it is and is not like $\text{Proj}_{nc}(S(E, \tau))$. Following the ideas and results in papers of Artin-Tate-Van den Bergh, Smith-Stafford, and Levasseur-Smith, we examine the line modules, point modules, and fat point modules, over A , and their incidence relations. Point modules and fat point modules determine “points” in $\text{Proj}_{nc}(A)$. Line modules determine “lines” in $\text{Proj}_{nc}(A)$.

For a moment, let A denote a 3-dimensional quadratic Artin-Schelter regular algebra such that $\text{Proj}_{nc}(A) \not\cong \mathbb{P}^2$, i.e., $\text{Proj}_{nc}(A)$ is a genuine non-commutative deformation of \mathbb{P}^2 . The “points” in $\text{Proj}_{nc}(A)$ form a closed subspace of $\text{Proj}_{nc}(A)$ that is a genuine commutative curve of arithmetic genus one. That curve, which is called the *point scheme* of A or $\text{Proj}_{nc}(A)$, is a fine moduli space for a moduli problem concerning the parametrization of “points” in $\text{Proj}_{nc}(A)$. There is a similar moduli problem for “lines” in $\text{Proj}_{nc}(A)$ and the *line scheme*, the solution to the moduli problem, is isomorphic to \mathbb{P}^2 . There is also a theorem saying that every “line” in $\text{Proj}_{nc}(A)$ meets the point scheme with multiplicity three. All these results, and the attendant definitions, appear in [4]. The point scheme for a 3-dimensional Sklyanin algebra $A(E, \tau)$ is the image of a closed immersion $E \rightarrow \text{Proj}_{nc}(A)$.

Non-commutative analogues of \mathbb{P}^3 exhibit a much greater variety of behaviors and we still don't understand the typical, or generic, behaviour of points, lines, and their incidence relations, in that setting. Various papers of Michaela Vanncliff and her co-authors illustrate some of this variety: [46], [47], [26], [48], [29], [27], [28], [40], [41], [8]. More examples can be found in [38] and [21].

1.6. It was shown in [34] that the point scheme for $\text{Proj}_{nc}(S(E, \tau))$ consists of a copy of E and 4 additional points. The point scheme is, in a natural way, a closed subscheme of $\mathbb{P}(S_1^*) \cong \mathbb{P}^3$. That closed subscheme is the natural copy of E (embedded as a quartic curve) in $\mathbb{P}(S_1^*) \cong \mathbb{P}^3$ and the 4 additional points are the vertices of the 4 singular quadrics that contain E . In [22], it was shown that the “lines” in $\text{Proj}_{nc}(S(E, \tau))$ are in natural bijection with the lines in $\mathbb{P}(S_1^*)$ that are secant to E , and these are the same lines as those that lie on the pencil of quadrics in $\mathbb{P}(S_1^*)$ that contain E . The incidence relations between the lines and points in $\text{Proj}_{nc}(S(E, \tau))$ are exactly the same as the incidence relations between the secant lines and the points of $E \sqcup \{4 \text{ vertices}\}$.

In addition to the points in $\text{Proj}_{nc}(S(E, \tau))$, there are *fat points* (see §1.15). When τ has infinite order, there are 4 fat points of multiplicity n for each integer $n \geq 2$ (the 4 “isolated” point

modules should be considered fat points of multiplicity 1) [35]. When $S(E, \tau)$ degenerates to a homogenization of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ the fat points degenerate to the finite dimensional simple $U_q(\mathfrak{sl}_2)$ -modules [11, §§5,6].

1.7. In contrast to what happens for $\text{Proj}_{nc}(S(E, \tau))$, Van den Bergh has shown there are non-commutative analogues of \mathbb{P}^3 that have only 20 points [43]. Although he showed that such examples exist, and that this behavior is in some sense typical, he did not produce explicit examples. Since then, Vancliff and her collaborators have produced a wealth of such examples (loc. cit.) including one for which the point scheme is a single point of multiplicity 20. Remarkably, there are examples with exactly $n < \infty$ points if and only if $n \in \{1, \dots, 20\} - \{2, 15, 17, 19\}$ [41].

By [9, Thm. 9.3], $\text{Proj}_{nc}(A(E, \tau))$ has exactly 20 points.

1.8. Lines in $\text{Proj}_{nc}(A(E, \tau))$ and, what are essentially the same things, line modules for $A(E, \tau)$ are in bijection with certain lines in $\mathbb{P}(A_1^*) \cong \mathbb{P}^3$ and therefore correspond to the closed points of a certain subscheme \mathbb{L} of the Grassmannian $\mathbb{G}(1, 3)$. Shelton-Vancliff call \mathbb{L} the *line scheme* for A .

In this paper we complete the classification of lines in $\text{Proj}_{nc}(A(E, \tau))$. We show that \mathbb{L} is the union of 7 reduced and irreducible components, 3 quartic elliptic space curves and 4 plane conics in the ambient Plücker \mathbb{P}^5 , and that $\deg(\mathbb{L}) = 20$. The union of the lines corresponding to the points on each elliptic curve is an elliptic scroll in $\mathbb{P}(A_1^*)$. Thus, the lines on that elliptic scroll are in natural bijection with a corresponding family of line modules for A .

The fact that the degree of \mathbb{L} is 20 is a special case of a general phenomenon: if $A = TV/(R)$ is an algebra generated by a 4-dimensional vector space V subject to a 6-dimensional subspace of relations $R \subseteq V^{\otimes 2}$, then every irreducible component of the line scheme \mathbb{L} has dimension ≥ 1 and if $\dim(\mathbb{L}) = 1$, then $\deg(\mathbb{L}) = 20$ [10, Thm. 1.1].

We also determine the incidence relations between lines, points, and fat points of multiplicity 2, in $\text{Proj}_{nc}(A(E, \tau))$.

1.9. It is pointed out in [27, Remark 3.2] that if a point in a non-commutative analogue of \mathbb{P}^3 lies on only finitely many lines, then it lies on exactly 6 lines counted with multiplicity. It is also remarked there that there is a paucity of such examples. We show that every point in $\text{Proj}_{nc}(A(E, \tau))$ lies on exactly 6 lines (Theorem 1.2 below).

1.10. The remainder of this introduction will make some of what we have just said more precise.

1.11. **The line scheme \mathbb{L} for A .** If A is any connected graded k -algebra generated by A_1 we call a module $L \in \text{Gr}(A)$ a *line module* if its Hilbert series is $(1 - t)^{-2}$ and $L = AL_0$.

In [27], Shelton and Vancliff formulate a moduli problem, “classification of line modules”, for certain graded algebras A and show there is a fine moduli space for it, the closed points of which are in natural bijection with the isomorphism classes of line modules for A . This moduli space is called the *line scheme* for A . We denote it by \mathbb{L} .

For $A = A(E, \tau)$, \mathbb{L} is a closed subscheme of the Grassmannian $\mathbb{G}(1, 3)$ that parametrizes the lines in $\mathbb{P}(A_1^*) \cong \mathbb{P}^3$. Often, we identify A_1 with S_1 and $\mathbb{P}(A_1^*)$ with $\mathbb{P}(S_1^*)$.

For each 2-torsion point ξ there is a natural bijection between the secant lines $\overline{p, p + \xi} \subseteq \mathbb{P}(S_1^*)$, $p \in E$, and isomorphism classes of certain line modules for A [9, §10].

For each 2-torsion point ξ the union of the secant lines $\overline{p, p + \xi}$ is an elliptic scroll in $\mathbb{P}(S_1^*)$ and there is a corresponding closed immersion $E/\langle \xi \rangle \rightarrow \mathbb{G}(1, 3)$. We frequently identify $E/\langle \xi \rangle$ with its image in $\mathbb{G}(1, 3)$.

The degree of a closed subscheme of $\mathbb{G}(1, 3)$ is the degree of its image under the Plücker embedding $\mathbb{G}(1, 3) \rightarrow \mathbb{P}^5$.

Theorem 1.1 (Theorem 7.1). *Let ξ_1, ξ_2, ξ_3 be the 2-torsion points of E . The line scheme for A is a reduced irreducible curve of degree 20. It is the union of 3 disjoint quartic elliptic curves, $E_j \cong E/\langle \xi_j \rangle$, and 4 disjoint plane conics,*

$$\mathbb{L} = (E_1 \sqcup E_2 \sqcup E_3) \bigcup (C_0 \sqcup C_1 \sqcup C_2 \sqcup C_3),$$

having the property that $|C_i \cap E_j| = 2$ for all $(i, j) \in \{0, 1, 2, 3\} \times \{1, 2, 3\}$.

We describe the line modules parametrized by the conics C_j in Sections 5 and 6. There we show that \mathbb{L} contains the components in Theorem 1.1. To show their union is \mathbb{L} we use a result by Derek Tomlin showing that $\dim(\mathbb{L}) \leq 1$. We are grateful to Derek Tomlin for writing an appendix to this paper in which he provides a computer calculation of the reduced scheme \mathbb{L}_{red} that we use to show that \mathbb{L} is contained in the union of the components in Theorem 1.1. His work will appear, with more detail, in his Ph.D. thesis [42]. The key point, for us, is that Tomlin shows that $\dim(\mathbb{L}) \leq 1$. This inequality, when $\text{char}(k) = 0$ and τ is “generic”, appeared earlier in the thesis of Andrew Davies [16, Thm. 6.1.27].

1.12. Points on C_0 correspond to lines $y = y' = 0$ in $\mathbb{P}(A_1^*)$ where y and y' are linearly independent elements in A_1 such that $[y, y'] = 0$. The conics C_1, C_2, C_3 , are obtained from C_0 by using auto-equivalences of the category of graded left A -modules, $\text{Gr}(A)$, that are induced by a coaction $A \rightarrow A \otimes H$ of a finite dimensional Hopf algebra H . We think of H as the coordinate ring of a finite quantum group acting on A . As a coalgebra, $H = k(H_4)$, the ring of k -valued functions on the Heisenberg group H_4 of order 4^3 . As an algebra, though, H is not commutative.

The isomorphism classes of the line modules parametrized by C_j are in natural bijection with the lines in a ruling on a smooth quadric in $\mathbb{P}(A_1^*)$, the equations of which appear in Table 5.

1.13. In [9], we found various A -modules that become irreducible in the quotient category

$$\text{QGr}(A) := \frac{\text{Gr}(A)}{\text{Fdim}(A)}$$

where $\text{Fdim}(A)$ is the full subcategory of $\text{Gr}(A)$ consisting of those graded modules that are the sum of their finite dimensional submodules. We write π^* for the quotient functor $\text{Gr}(A) \rightarrow \text{QGr}(A)$ and π_* for its right adjoint.

The irreducible objects in $\text{QGr}(A)$ that we found in [9] were of the form π^*N where N was either a point module for A or a fat point module for A of multiplicity two.

We think of $\text{QGr}(A)$ as if it is the category of quasi-coherent sheaves on the implicitly defined non-commutative variety $\text{Proj}_{nc}(A)$. We think of $\text{Proj}_{nc}(A)$ as a non-commutative analogue of \mathbb{P}^3 . The geometric properties of $\text{Proj}_{nc}(S)$ and $\text{Proj}_{nc}(A)$ are non-commutative analogues of the very beautiful geometric properties of quartic elliptic curves in \mathbb{P}^3 and our main results are best understood in that context.

1.14. For both S and A there is a non-commutative analogue of the fact that E is the base locus of a pencil of quadrics in $\mathbb{P}(S_1^*)$.

By [34], S has a central subalgebra $k[\Omega, \Omega']$ that is a polynomial ring in two variables $\Omega, \Omega' \in S_2$ and “the zero locus of (Ω, Ω') in $\text{Proj}_{nc}(S)$ is isomorphic to E ” in the sense that $S/(\Omega, \Omega')$ is isomorphic to a twisted homogeneous coordinate ring $B(E, \tau, \mathcal{L})$ and $\text{QGr}(B(E, \tau, \mathcal{L}))$ is equivalent to $\text{Qcoh}(E)$ by [5]. We say that $\text{Proj}_{nc}(S)$ contains a copy of E because the homomorphism $S \rightarrow B(E, \tau, \mathcal{L})$ induces a fully faithful functor $i_* : \text{Qcoh}(E) \rightarrow \text{QGr}(S)$ that has a left and a right adjoint and the essential image of i_* is closed under subquotients. In other words, i_* behaves like a direct image functor for a closed immersion $E \rightarrow \text{Proj}_{nc}(S)$.

By [9], A also has a central subalgebra $k[\Theta, \Theta']$ that is a polynomial ring in two variables $\Theta, \Theta' \in A_2$ and “the zero locus of (Θ, Θ') in $\text{Proj}_{nc}(A)$ is isomorphic to $E/E[2]$ ” in the sense

that $\mathrm{QGr}(A/(\Theta, \Theta'))$ is equivalent to $\mathrm{Qcoh}(E/E[2])$. The ring $A/(\Theta, \Theta')$ is not a twisted homogeneous coordinate ring in the sense of [5]. The natural functor $\mathrm{Gr}(A/(\Theta, \Theta')) \rightarrow \mathrm{Gr}(A)$ induces a functor $\mathrm{Qcoh}(E/E[2]) \rightarrow \mathrm{QGr}(A)$ that behaves like the direct image functor for a “closed immersion” $E/E[2] \rightarrow \mathrm{Proj}_{nc}(A)$. Although $E/E[2]$ is isomorphic to E it is better to think of it as $E/E[2]$ in this context.

1.15. Points, lines, fat points, and quadrics. A module $N \in \mathrm{Gr}(A)$ is a point module if its Hilbert series is $(1-t)^{-1}$ and $N = AN_0$. By [9, Thm. 9.3], the set \mathfrak{P} of isomorphism classes of point modules has cardinality 20, and is a disjoint union of five 4-element subsets that we labelled $\mathfrak{P}_\infty, \mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3$. We call point modules in \mathfrak{P}_∞ special and those in the other \mathfrak{P}_j ordinary.

A module $F \in \mathrm{Gr}(A)$ is a fat point module of multiplicity two if its Hilbert series is either $2(1-t)^{-1}$ or $(1+t)(1-t)^{-1}$, $F = AF_0$, and π^*F is a simple object in $\mathrm{QGr}(A)$ (the last condition is equivalent to the condition that every proper quotient of F has finite dimension). The image in $\mathrm{QGr}(A)$ under the “direct image functor” of the skyscraper sheaf $\mathcal{O}_{p+E[2]}$ at a point $p+E[2] \in E/E[2]$ is a simple object in $\mathrm{QGr}(A)$ and the truncation $(\pi_*\mathcal{O}_{p+E[2]})_{\geq 0}$ is a fat point of multiplicity two with Hilbert series $2(1-t)^{-2}$. If v is any non-zero element in $(\pi_*\mathcal{O}_{p+E[2]})_0$, then Av is a fat point of multiplicity two with Hilbert series $(1+t)(1-t)^{-2}$. See [9, Thm. 8.1 and Cor. 8.7].

If $N \in \mathrm{Gr}(A)$ is a point module (resp., a fat point module) we will often refer to the isomorphism class of π^*N in $\mathrm{QGr}(A)$ as a *point* (resp., a *fat point of multiplicity two*) because it is the analogue of the structure sheaf of a point (resp., a point with multiplicity two).

If L is a line module for A we call the isomorphism class of π^*L a *line* because it is the analogue of the structure sheaf of a line.

Since isomorphism classes of line modules and point modules for A , or S , are in bijection with certain lines and points in $\mathbb{P}(A_1^*)$, or $\mathbb{P}(S_1^*)$, there is some potential for confusion when one speaks of a line or point. We will be careful to avoid ambiguity.

1.16. Incidence relations. After we classify the line modules for A , we determine which points and fat points of multiplicity two lie on which lines. We say that a point π^*N , or a fat point π^*F , lies on the line π^*L if there is an epimorphism $\pi^*L \rightarrow \pi^*N$ or $\pi^*L \rightarrow \pi^*F$. We also express this by saying that the point π^*N , or the fat point π^*F , belongs to the line π^*L . We note that π^*N , or π^*F , lies on π^*L if and only if there is a surjective homomorphism $L_{\geq n} \rightarrow N_{\geq n}$, or $L_{\geq n} \rightarrow F_{\geq n}$ for $n \gg 0$.

Theorem 1.2.

- (1) Every point in \mathfrak{P}_∞ lies on exactly two lines in each $E/\langle \xi \rangle$ and lies on no lines in $C_0 \cup \dots \cup C_3$.
- (2) Every point in \mathfrak{P}_j , $j \neq \infty$, lies on exactly one line in each $E/\langle \xi \rangle$, on exactly one line in C_i if $i \neq j$, and on no lines in C_j .
- (3) Every fat point of multiplicity two corresponding to a point in $E/E[2]$ lies on exactly two lines in each family $E/\langle \xi \rangle$.
- (4) For $i = 0, 1, 2, 3$, there is a unique fat point of multiplicity two in $E/E[2]$ through which every line in C_i passes, and ... see below.

To state (4) more precisely we must introduce the point

$$\tau' := (abc, a, b, c) \in E$$

which satisfies $2\tau' = -\tau$. In the previous sentence, a, b, c , are related to the structure constants for S and A (see §2.1). If $k = \mathbb{C}$, then a, b, c , are the values at τ of certain meromorphic functions on E and, as a consequence of Jacobi’s theta function identity $\theta_{00}(\tau)^4 + \theta_{11}(\tau)^4 = \theta_{01}(\tau)^4 + \theta_{10}(\tau)^4$, they satisfy the identity $a^2b^2c^2 + a^2 + b^2 + c^2 = 0$. Now we can state (4) more precisely:

- (4) there are 4-torsion points $o = \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$, such that every line in C_i passes through the fat point $\tau' + \varepsilon_i$ of multiplicity two and through no other fat point of multiplicity two in $E/E[2]$. In coordinates, $\tau' + \varepsilon_1 = (a, -ia, i, 1)$, $\tau' + \varepsilon_2 = (b, 1, -ib, i)$, $\tau' + \varepsilon_3 = (c, i, 1, -ic)$.

Our understanding of [Theorem 1.2\(4\)](#) is informed by the fact that there are exactly four singular quadrics in $\mathbb{P}(S_1^*)$ that contain E and the vertex of each singular quadric lies on every line on that quadric. One can embed $E/E[2]$ as a quartic curve in \mathbb{P}^3 in such a way that 4 points of $E/E[2]$ are coplanar if and only if their sum is $o + E[2]$. There is a pencil of quadrics that contain $E/E[2]$. The surfaces in this pencil may be labeled Q_z , $z \in E/E[2]$, in such a way that $Q_z = Q_{-z}$ and the secant line \overline{pq} through $p, q \in E/E[2]$ lies on Q_z if and only if $p + q = \pm z$. There are exactly 4 singular quadrics containing $E/E[2]$, namely $\{Q_{\varepsilon + E[2]} \mid \varepsilon \in E[4]/E[2]\}$. Thus, if $\varepsilon \in E[4]/E[2]$, then all lines in the pencil of secant lines $\{\overline{pq} \mid p + q = \varepsilon + E[2]\}$ pass through the vertex of Q_ε . We think of [Theorem 1.2\(4\)](#) as saying that $\tau' + \varepsilon_i + E[2]$ behaves like the vertex of one of these singular quadrics in so far as all the lines (in $\text{Proj}_{nc}(A)$) parametrized by C_i pass through $\tau' + \varepsilon_i + E[2]$.

We use the adjectives *elliptic* and *conic* for lines and/or line modules parametrized by the E_j 's and the C_i 's, respectively.

1.17. New results on the Sklyanin algebra. Although A is our primary interest in this paper we need some new results on the 4-dimensional Sklyanin algebras. These results are in [§2](#).

We determine four explicit two-dimensional irreducible representations $V(\tau + \xi)$, $\xi \in E[2]$, of S whose “homogenizations”, $\tilde{V}(\tau + \xi)$, are fat points of multiplicity 2; i.e., $\dim_k(\tilde{V}(\tau + \xi)_n) = 2$ when $n \geq 0$ and is zero otherwise, and every proper quotient of $\tilde{V}(\tau + \xi)$ has finite dimension. Previously, graded modules with these properties were constructed in [\[35\]](#) only when $k = \mathbb{C}$ and then in terms of theta functions.

We use the modules $\tilde{V}(\tau + \xi)$ to provide a better, more abstract, description of the point modules for A . As a module over $S \otimes M_2(k)$, each $\tilde{V}(\tau + \xi) \otimes k^2$ has four different Γ -equivariant structures and the Γ -invariant subspaces give four different point modules for A . This equivariant construction of the point modules for A enables us to determine which elliptic line modules map onto them. The elliptic line modules for A are obtained in [\[9, §11\]](#) by a similar equivariant construction.

In [§2](#), and in later sections, we exploit the action of the Heisenberg group of order 4^3 as automorphisms of S .

1.18. Non-commutative quadrics in $\text{Proj}_{nc}(A)$. If z is a non-zero degree-two homogeneous central element in A it is reasonable to view $\text{QGr}(A/(z))$ as the category of “quasi-coherent sheaves” on a “non-commutative quadric” $\text{Proj}_{nc}(A/(z)) \subseteq \text{Proj}_{nc}(A)$. A justification for this view can be found in sections 10 and 11 of Stafford and Van den Bergh’s survey [\[39\]](#), and in [\[36\]](#).

The best understood non-commutative surfaces are those that are analogous to \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$. Non-commutative quadrics, meaning non-commutative analogues of $\mathbb{P}^1 \times \mathbb{P}^1$, are also treated in some detail in [\[36\]](#) and [\[45\]](#). We don’t understand how $\text{Proj}_{nc}(A/(z))$ fits into the taxonomy of non-commutative surfaces. For example, we do not know if $\text{Proj}_{nc}(A/(z))$ is birationally isomorphic to any previously known non-commutative surfaces.

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2. THE 4-DIMENSIONAL SKLYANIN ALGEBRAS $S(E, \tau)$

2.1. Definition of the Sklyanin algebra. Always, $\alpha_1, \alpha_2, \alpha_3$ are fixed elements in $k - \{0, \pm 1\}$ such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$. We often write $\alpha = \alpha_1$, $\beta = \alpha_2$, and $\gamma = \alpha_3$. As in [\[9, §6\]](#), we fix $a, b, c, i \in k$ such that $a^2 = \alpha$, $b^2 = \beta$, $c^2 = \gamma$, and $i^2 = -1$.

The 4-dimensional Sklyanin algebra, $S = S(\alpha_1, \alpha_2, \alpha_3)$, is the quotient of the free algebra $k\langle x_0, x_1, x_2, x_3 \rangle$ by the six relations

$$(2-1) \quad x_0x_i - x_ix_0 = \alpha_i(x_jx_k + x_kx_j), \quad x_0x_i + x_ix_0 = x_jx_k - x_kx_j,$$

where (i, j, k) runs over the cyclic permutations of $(1, 2, 3)$.

2.1.1. *Notation.* If r and s are elements in a ring R we write $[r, s]$ for their commutator and $\{r, s\}$ for their anti-commutator. With this notation, the relations for S are

$$[x_0, x_i] - \alpha_i\{x_j, x_k\} = \{x_0, x_i\} - [x_j, x_k] = 0.$$

2.2. **The curve** $E \subseteq \mathbb{P}^3 = \mathbb{P}(S_1^*)$. Sklyanin originally defined S over \mathbb{C} and did so by defining $\alpha_1, \alpha_2, \alpha_3$ to be values of certain meromorphic functions on an elliptic curve, E , evaluated at a particular point $\tau \in E$. Later, in [34, §2.4], it was shown that there is a “natural” copy of E in $\mathbb{P}(S_1^*)$ embedded as the quartic curve that is the intersection of any two of the quadrics

$$(2-2) \quad \begin{cases} x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \\ x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2 = 0, \\ x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2 = 0, \\ x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2 = 0. \end{cases}$$

These formulas are illuminated by the calculation in (2-5) and by Proposition 2.15.

Thus, E is the base locus of a pencil of quadrics. Exactly four of these quadrics are singular. We label the vertices of the singular quadrics e_0, \dots, e_3 in such a way that e_j is the point where $\{x_0, \dots, x_3\} - \{x_j\}$ vanish. For example, $e_1 = (0, 1, 0, 0)$.

Let $p, q \in E$. We write \overline{pq} for the line in $\mathbb{P}(S_1^*)$ whose scheme-theoretic intersection with E is the divisor $(p) + (q)$.

Lemma 2.1. *The quadrics containing E may be labelled $Q(z)$, $z \in E$, in such a way that*

- (1) $Q(z) = Q(-z)$;
- (2) $Q(z)$ is singular if and only if $z \in E[2]$;
- (3) the lines \overline{pq} , $p + q = z$, provide a ruling on $Q(z)$.

Proof. Given $z \in E$, define $Q(z)$ to be the union of the secant lines $\overline{p, z-p}$ as p ranges over E .

As remarked in §1.3, the group law on E is chosen so it has the following property: if $p, q, r, s \in E$, then $p+q+r+s = o$ if and only if there is a plane $H \subseteq \mathbb{P}^3$ such that the scheme-theoretic intersection $H \cap E$ is the divisor $(p) + (q) + (r) + (s)$.

Let Q be a smooth quadric containing E . Let ℓ, ℓ' , and ℓ'' be lines on Q such that ℓ' and ℓ'' belong to the same ruling and ℓ belongs to the other ruling. Let $p, p', p'', q, q', q'' \in E$ be such that the scheme-theoretic intersections are $\ell \cap E = (p) + (q)$, $\ell' \cap E = (p') + (q')$, and $\ell'' \cap E = (p'') + (q'')$. Since $\ell \cap \ell' \neq \emptyset$, there is a plane H containing $\ell \cup \ell'$; since $H \cap E = (p) + (q) + (p') + (q')$, $p + q + p' + q' = o$. Similarly, $p + q + p'' + q'' = o$. Thus, if $z = p + q$, then $Q = Q(z) = Q(-z)$. Since Q has two different rulings on it, $z \neq -z$, i.e., $z \notin E[2]$.

Now let Q be a singular quadric containing E . Let $\ell = \overline{pq}$ and $\ell' = \overline{p'q'}$ be different lines on Q . Since ℓ and ℓ' meet at the vertex of Q , ℓ and ℓ' lie on a common plane that meets E at $(p) + (q) + (p') + (q')$. Hence $p + q + p' + q' = o$. Since Q is singular, there is a plane that meets Q in ℓ with multiplicity two. Hence $p + q + p + q = o$. Thus, if $z = p + q$, then $z \in E[2]$ and $p' + q' = z = -z$. Thus, $Q = Q(z) = Q(-z)$.

The result follows. ■

We will label the elements in $E[2] = \{o = \xi_0, \xi_1, \xi_2, \xi_3\}$ are labelled so that e_j is the vertex of the quadric $Q(\xi_j)$.

In Proposition 2.15, we will show that the four quadrics cut out by the equations in (2-2) are $Q(\tau + \xi)$, $\xi \in E[2]$.

2.3. Central elements in S . There is a non-commutative analogue of the fact that the image of E in $\mathbb{P}(S_1^*)$ is the intersection of a pencil of quadrics: there is a closed immersion $E \rightarrow \text{Proj}_{nc}(S)$ such that the image of E is the “intersection of a pencil of non-commutative quadrics”. (See §1.14.)

Proposition 2.2. (cf., [34, Cor. 3.9], [22, p.39]) *The center of S contains $\Omega = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ and the elements*

$$(2-3) \quad \begin{cases} \Omega_0 := (1 + \gamma)x_1^2 + (1 + \alpha\gamma)x_2^2 + (1 - \alpha)x_3^2, \\ \Omega_1 := (1 + \beta\gamma)x_0^2 - (\gamma + \beta\gamma)x_2^2 + (\beta - \beta\gamma)x_3^2, \\ \Omega_2 := (1 + \alpha\gamma)x_0^2 + (\gamma - \alpha\gamma)x_1^2 - (\alpha + \alpha\gamma)x_3^2, \\ \Omega_3 := (1 + \alpha\beta)x_0^2 - (\beta + \alpha\beta)x_1^2 + (\alpha - \alpha\beta)x_2^2. \end{cases}$$

These elements satisfy the relations $2\alpha\beta\gamma\Omega_0 + \alpha\Omega_1 + \beta\Omega_2 + \gamma\Omega_3 = 0$ and

$$\alpha_j(1 + \alpha_i)(1 + \alpha_j)\Omega = (1 + \alpha_i\alpha_j)\Omega_i - (1 + \alpha_j)\Omega_k$$

if (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Proof. By [34, Cor. 3.9], Ω and $\Omega' := x_1^2 + \left(\frac{1+\alpha}{1-\beta}\right)x_2^2 + \left(\frac{1-\alpha}{1+\gamma}\right)x_3^2$ belong to the center of S . Error-prone computations show that the elements in (2-3) are linear combinations of Ω and Ω' so belong to the center of S . The first step in the proof is to observe that $(1 + \gamma)\Omega' = \Omega_0$ because $(1 + \alpha)(1 + \gamma) = (1 - \beta)(1 + \alpha\gamma)$. ■

2.4. Line modules for S and their annihilators. Let $p, q \in E$. We write $(\overline{pq})^\perp$ for the 2-dimensional subspace of S_1 that vanishes on \overline{pq} .

Let $S^2E := E \times E / \sim$ where \sim is the equivalence relation $(p, q) \sim (q, p)$. The closed points in S^2E are in bijection with the effective divisors of degree 2: $(p) + (q)$ is the image in S^2E of (p, q) .

Theorem 2.3. [22, Thm. 4.5] *The function $S^2E \rightarrow \text{Gr}(S)$ that sends $(p) + (q)$ to $S/S(\overline{pq})^\perp$ is a bijection from S^2E to the set of isomorphism classes of line modules for S .*

If $p, q \in E$, we define $M_{p,q} := S/S(\overline{pq})^\perp$.

2.4.1. The central elements $\Omega(z)$. Let $z \in E$. We will abuse notation and use the symbol $\Omega(z)$ to denote any non-zero element in $k\Omega_0 + k\Omega_1$ that annihilates all line modules $M_{p,q}$ for which $p + q = z$. By [22], $\Omega(z) = \Omega(-z - 2\tau)$. Since $\Omega(z)$ is only defined up to a non-zero scalar multiple, the previous sentence has the same meaning as the phrase “a particular line module is annihilated by $\Omega(z)$ if and only if it is annihilated by $\Omega(-z - 2\tau)$ ”.

Proposition 2.4. [22, Lem. 6.2, Cor. 6.6, Prop. 6.4, Prop. 6.8] *Let $p, q \in E$.*

- (1) *The line module $M_{p,q}$ is annihilated by $\Omega(z)$ if and only if $p + q \in \{z, -z - 2\tau\}$.*
- (2) *The line \overline{pq} lies on one of the singular quadrics that contains E if and only if $p + q \in E[2]$.*

2.5. Point modules for S . If $p \in \mathbb{P}(S_1^*)$, we write p^\perp for the subspace of S_1 that vanishes at p .

Theorem 2.5. [34] *If e_0, \dots, e_3 are the vertices of the singular quadrics that contain E , then*

$$\{\text{isomorphism classes of point modules for } S\} = \{S/Sp^\perp \mid p \in E \sqcup \{e_0, e_1, e_2, e_3\}\}.$$

If $p \in E \sqcup \{e_0, e_1, e_2, e_3\}$, we define $M_p := S/Sp^\perp$.

Theorem 2.6. [22, Thms. 5.5, 5.7] *Let $p, q \in E$.*

- (1) *There is an exact sequence $0 \rightarrow M_{p+\tau, q-\tau}(-1) \rightarrow M_{p,q} \rightarrow M_p \rightarrow 0$.*
- (2) *If $p + q = \xi_j$, then there is an exact sequence $0 \rightarrow M_{p-\tau, q-\tau}(-1) \rightarrow M_{p,q} \rightarrow M_{e_j} \rightarrow 0$.*
- (3) *M_{e_j} is annihilated by the central element $\Omega_j = \Omega(\xi_j)$ in (2-3).*

We call the point modules corresponding to the e_i ’s **special**.

One reason point modules are important is that they are irreducible/simple as objects in the quotient category $\text{QGr}(S)$. Not all simple objects in $\text{QGr}(S)$ arise from point modules (see §2.10).

2.6. The action of Γ and $E[2]$ as automorphisms of S . In [9, §6], we defined an action of the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1 = \gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ as k -algebra automorphisms of S . Table 1 describes the action: the entry in row γ_i and column x_j is $\gamma_i(x_j)$.

	x_0	x_1	x_2	x_3
γ_1	x_0	x_1	$-x_2$	$-x_3$
γ_2	x_0	$-x_1$	x_2	$-x_3$
γ_3	x_0	$-x_1$	$-x_2$	x_3

TABLE 1. The action of Γ and $E[2]$ as automorphisms of S .

The induced action of Γ on $\mathbb{P}(S_1^*)$ restricts to an action of Γ as automorphisms of E . The results in [9, §7] showed that if $p \in E \subseteq \mathbb{P}(S_1^*)$, then $\gamma_j(p) = p + \xi_j$. Because $\gamma_j(p) = p + \xi_j$ we often blur the distinction between Γ and $E[2]$ although we usually use multiplicative notation for Γ and additive notation for $E[2]$.

2.6.1. Let A be a \mathbb{Z} -graded k -algebra. Let $\text{Aut}_{\text{gr}}(A)$ denote the group of graded k -algebra automorphisms of A . If $\lambda \in k^\times$, let ϕ_λ be the automorphism of A that is multiplication by λ^n on A_n . Then $\lambda \mapsto \phi_\lambda$ is a homomorphism $k^\times \rightarrow \text{Aut}_{\text{gr}}(A)$. We will often identify $\lambda \in k^\times$ with ϕ_λ and, if $\psi \in \text{Aut}(A)$, we will write $\psi^m = \lambda$ if $\psi^m = \phi_\lambda$ and $\lambda\psi$ for $\phi_\lambda\psi$.

2.6.2. If ψ is a k -algebra automorphism of a k -algebra R and M is a left R -module we write ψ^*M for the left R -module that is M as a vector space with the new action given by $x \cdot m := \psi(x)m$.

2.7. The action of the Heisenberg group H_4 as automorphisms of S . By [19] and [35, pp. 64-65], for example, the Heisenberg group of order 4^3 acts as graded k -algebra automorphisms of S when $k = \mathbb{C}$. The next result shows this holds without restriction on k .

Proposition 2.7. *If (i, j, k) is a cyclic permutation of $(1, 2, 3)$, then there is a k -algebra automorphism ϕ_i of S such that $\phi_i(x_0) = a_j a_k x_i$, $\phi_i(x_i) = -ix_0$, $\phi_i(x_j) = -ia_j x_k$, and $\phi_i(x_k) = -a_k x_j$, where $(a_1, a_2, a_3) = (a, b, c)$. Thus, $\phi_i(x_j)$ is the entry in row ϕ_i and column x_j in Table 2.*

	x_0	x_1	x_2	x_3
ϕ_1	bcx_1	$-ix_0$	$-ibx_3$	$-cx_2$
ϕ_2	acx_2	$-ax_3$	$-ix_0$	$-icx_1$
ϕ_3	abx_3	$-iax_2$	$-bx_1$	$-ix_0$

TABLE 2. Automorphisms of S .

Fix $\nu_1, \nu_2, \nu_3 \in k^\times$ such that $a\nu_1^2 = b\nu_2^2 = c\nu_3^2 = -iabc$. Let $\varepsilon_1 = \nu_1^{-1}\phi_1$, $\varepsilon_2 = \nu_2^{-1}\phi_2$, $\varepsilon_3 = \nu_3^{-1}\phi_3$, and $\delta = i$. The subgroup $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \delta \rangle \subseteq \text{Aut}(S)$ is isomorphic to the Heisenberg group of order 4^3 ,

$$H_4 := \langle \varepsilon_1, \varepsilon_2, \delta \mid \varepsilon_1^4 = \varepsilon_2^4 = \delta^4 = 1, \delta\varepsilon_1 = \varepsilon_1\delta, \varepsilon_2\delta = \delta\varepsilon_2, \varepsilon_1\varepsilon_2 = \delta\varepsilon_2\varepsilon_1 \rangle.$$

Furthermore, $\varepsilon_1^2 = \gamma_1$, $\varepsilon_2^2 = \gamma_2$, and $\varepsilon_3^2 = \gamma_3$ where $\gamma_1, \gamma_2, \gamma_3$ are the automorphisms of S in Table 1. In particular, we can identify Γ with the subgroup $\langle \varepsilon_1^2, \varepsilon_2^2 \rangle \subseteq H_4 \subseteq \text{Aut}(S)$.

Proof. Let (i, j, k) be a cyclic permutation of $(1, 2, 3)$, let $\lambda_0, \lambda_i, \lambda_j, \lambda_k \in k^\times$, and let $\phi : S_1 \rightarrow S_1$ be the linear map acting on x_0, x_i, x_j, x_k as

	x_0	x_i	x_j	x_k
ϕ	$\lambda_0 x_i$	$\lambda_i x_0$	$\lambda_j x_k$	$\lambda_k x_j$

Following [31, Prop. 4], it is easy to see that ϕ extends to an automorphism of S if and only if

$$\frac{\lambda_0 \lambda_i}{\lambda_j \lambda_k} = -1, \quad \frac{\lambda_0 \lambda_j}{\lambda_k \lambda_i} = -\alpha_j, \quad \text{and} \quad \frac{\lambda_0 \lambda_k}{\lambda_i \lambda_j} = \alpha_k.$$

The maps ϕ_1 , ϕ_2 , and ϕ_3 , satisfy these conditions so extend to automorphisms of S .

Simple calculations show that $\phi_1^2 = -ibc\gamma_1$, $\phi_2^2 = -iac\gamma_2$, and $\phi_3^2 = -iab\gamma_3$, where γ_1 , γ_2 , and γ_3 , are the automorphisms in Table 1. (For the sake of symmetry, let $\varepsilon_3 = \nu_3^{-1}\phi_3$.) It follows that $\varepsilon_1^2 = \gamma_1$, $\varepsilon_2^2 = \gamma_2$, and $\varepsilon_3^2 = \gamma_3$. Hence $\varepsilon_1^4 = \varepsilon_2^4 = \varepsilon_3^4 = 1$. It is easy to check that $\phi_1\phi_2 = i\phi_2\phi_1$ which implies that $\varepsilon_1\varepsilon_2 = i\varepsilon_2\varepsilon_1 = \delta\varepsilon_2\varepsilon_1$. We leave the rest of the proof to the reader. ■

2.7.1. *Remark.* Later we will make use of the fact that $\varepsilon_2\varepsilon_3 = \delta\varepsilon_3\varepsilon_2$ and $\varepsilon_3\varepsilon_1 = \delta\varepsilon_1\varepsilon_3$. These equalities follow from the fact that $\phi_2\phi_3 = \delta\phi_3\phi_2$ and $\phi_3\phi_1 = \delta\phi_1\phi_3$.

Proposition 2.8. H_4 acts on the central element $\Omega = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ as follows:

$$(2-4) \quad \begin{cases} -\phi_1(\Omega) = \Omega_1 + \beta\gamma\Omega = x_0^2 + \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2, \\ -\phi_2(\Omega) = \Omega_2 + \alpha\gamma\Omega = x_0^2 + \gamma x_1^2 + \alpha\gamma x_2^2 - \alpha x_3^2, \\ -\phi_3(\Omega) = \Omega_3 + \alpha\beta\Omega = x_0^2 - \beta x_1^2 + \alpha x_2^2 + \alpha\beta x_3^2. \end{cases}$$

2.7.2. The elements of S in (2-4) look much like the polynomials in (2-2) that cut out E .

2.7.3. **Proposition 2.7** gives a representation of H_4 on S_1 . We let H_4 act on S_1^* via the contragredient representation.

The center of H_4 is $\langle \delta \rangle$ and $H_4/\langle \delta \rangle \cong (\mathbb{Z}/4)^2$. The induced action of H_4 as automorphisms of $\mathbb{P}(S_1^*)$ factors through $H_4/\langle \delta \rangle$ thereby giving an action of $(\mathbb{Z}/4)^2$ as automorphisms of $\mathbb{P}(S_1^*)$.

Proposition 2.9. *The action of H_4 on $\mathbb{P}(S_1^*)$ restricts to an action on E as translation by $E[4]$. Furthermore, if we identify $\varepsilon_j \in H_4$ with the 4-torsion point $\varepsilon_j(o)$, then $2\varepsilon_j = \xi_j$.*

Proof. Each $\varepsilon_j : S_1 \rightarrow S_1$ is a scalar multiple of the linear automorphism ϕ_j in **Proposition 2.7**. Thus, to show that E is stable under the action of ε_j it suffices to treat ϕ_j as an automorphism of the commutative polynomial ring $k[x_0, x_1, x_2, x_3]$ and show that the linear span of the polynomials in (2-2) is stable under the action of ϕ_j . Straightforward calculations show that

$$(2-5) \quad \phi_j(x_0^2 + x_1^2 + x_2^2 + x_3^2) = \begin{cases} -x_0^2 + \beta\gamma x_1^2 + \gamma x_2^2 - \beta x_3^2 & \text{if } j = 1, \\ -x_0^2 - \gamma x_1^2 + \alpha\gamma x_2^2 + \alpha x_3^2 & \text{if } j = 2, \\ -x_0^2 + \beta x_1^2 - \alpha x_2^2 + \alpha\beta x_3^2 & \text{if } j = 3. \end{cases}$$

These are scalar multiples of the other polynomials in (2-2). Similarly,

$$\phi_j(x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2) = \begin{cases} \beta\gamma(x_0^2 + x_1^2 + x_2^2 + x_3^2) & \text{if } j = 1, \\ \gamma(x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2) & \text{if } j = 2, \\ -\beta(x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2) & \text{if } j = 3. \end{cases}$$

These are also scalar multiples of the polynomials in (2-2). Hence E is stable under the actions of ε_1 , ε_2 , and ε_3 .

Since every automorphism of E as an algebraic variety is the composition of a translation and an automorphism of the group $(E, +, o)$, there is a point $q \in E$ and an automorphism ϕ of the group $(E, +)$ such that $\varepsilon_1(p) = \phi(p) + q$ for all $p \in E$. By **Proposition 2.7**, $\varepsilon_1^2 = \gamma_1$. By [9, §7], $\gamma_1(p) = p + \xi_1$. Thus, $p + \xi_1 = \phi^2(p) + \phi(q) + q$ for all $p \in E$. It follows that $p = \phi^2(p)$ and $\xi_1 = \phi(q) + q$.

Because the characteristic of k is not 2 or 3 the group of automorphisms of the group $(E, +)$ is cyclic of order 6 if $j(E) = 0$, cyclic of order 4 if $j(E) = 12^3$, and cyclic of order 2 in all other cases. Thus, in all cases there is a unique automorphism of $(E, +)$ having order 2, namely $p \mapsto -p$. See [20, Chap. 3], for example.

If ϕ is not the identity map, then $\phi(p) = -p$ for all $p \in E$ which implies that $p + \xi_1 = -p$ for all p . That is absurd, so we conclude that ϕ is the identity morphism and $p + \xi_1 = p + 2q$. Thus, ε_1 is translation by q which is a point of order 4. Similar arguments apply to ε_2 and ε_3 .

Since $\varepsilon_j^2 = \gamma_j$, and γ_j acts on E as translation by ξ_j , in E we have $\varepsilon_j + \varepsilon_j = \xi_j$. ■

As in the previous proof we will often identify the image of H_4 in $\text{Aut}(E)$ with the subgroup $E[4]$ acting by translation and when we do that we will identify the automorphism ε_j with the point $\varepsilon_j(o)$ which we will also label ε_j . For the sake of symmetry we set $\varepsilon_0 = o$.

Corollary 2.10. *The action of ε_j on S sends $\Omega(z)$ to $\Omega(z + \xi_j)$. In particular, up to scalar multiples, $\Omega(o) = \Omega_0$, and $\varepsilon_j(\Omega_0) = \Omega_j = \Omega(\xi_j)$.*

Proof. A straightforward calculation using the definitions of ε_j and Ω_0 shows that $\varepsilon_j(\Omega_0)$ is a scalar multiple of Ω_j for $j = 1, 2, 3$.

If $p, q \in E$, then the line module $M_{p,q}$ is annihilated by $\Omega(z)$ if and only if $p + q + \tau = \pm(z + \tau)$. The same idea as in the proof of [9, Prop. 7.7], shows that the auto-equivalence ε_j^* of $\text{Gr}(S)$ induced by the automorphism ε_j has the property that $\varepsilon_j^*(M_{p,q}) \cong M_{p+\varepsilon_j, q+\varepsilon_j}$ where $p + \varepsilon_j$ denotes translation by the 4-torsion point ε_j . Since $\Omega(z + 2\varepsilon_j)$ annihilates $M_{p+\varepsilon_j, q+\varepsilon_j}$, we conclude that the automorphism ε_j sends $\Omega(z)$ to $\Omega(z + \xi_j)$. ■

Proposition 2.11. *With respect to the coordinate functions (x_0, x_1, x_2, x_3) on S_1^* and $\mathbb{P}(S_1^*)$, the action of $\varepsilon_j \in H_4$ on a point $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{P}(S_1^*)$ is*

$$(2-6) \quad \varepsilon_j(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \begin{cases} (bc\lambda_1, -i\lambda_0, ib\lambda_3, c\lambda_2) & \text{if } j = 1, \\ (ac\lambda_2, a\lambda_3, -i\lambda_0, ic\lambda_1) & \text{if } j = 2, \\ (ab\lambda_3, ia\lambda_2, b\lambda_1, -i\lambda_0) & \text{if } j = 3. \end{cases}$$

By Proposition 2.9, the formulas in (2-6) have the following interpretation.

Corollary 2.12. *Let $\varepsilon_j \in E[4]$, $j = 1, 2, 3$, be the points $\varepsilon_j(o)$. If $p = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in E$, then*

$$\begin{aligned} p + \varepsilon_1 &= (bc\lambda_1, -i\lambda_0, ib\lambda_3, c\lambda_2), \\ p + \varepsilon_2 &= (ac\lambda_2, a\lambda_3, -i\lambda_0, ic\lambda_1), \\ p + \varepsilon_3 &= (ab\lambda_3, ia\lambda_2, b\lambda_1, -i\lambda_0). \end{aligned}$$

One can use the formulas in Corollary 2.12 to check that $p + 2\varepsilon_j = \gamma_j(p)$ where γ_j is the automorphism of $\mathbb{P}(S_1^*)$ induced by the linear automorphism γ_j of S_1 in Table 1.

Corollary 2.13. *The 4-torsion subgroup $E[4]$ is the intersection of E with the four “coordinate” planes $x_0x_1x_2x_3x_4 = 0$. More precisely, if $j \in \{0, 1, 2, 3\}$, then*

$$\varepsilon_j + E[2] = E \cap \{x_j = 0\}.$$

Proof. By [9, §7], $E[2]$ is the intersection of E with the plane $x_0 = 0$. It follows from the formulas in Corollary 2.12 that $\varepsilon_j + E[2] = E \cap \{x_j = 0\}$. ■

2.8. The points in $-\frac{1}{2}\tau + E[2]$ and the quadrics $Q(\tau + \xi)$, $\xi \in E[2]$. The point

$$\tau' := (abc, a, b, c) \in E$$

plays an important role in the rest of §2 and in §11.

Proposition 2.14. *Let $\tau' = (abc, a, b, c)$. Then $2\tau' = -\tau$, $-\tau' = (-abc, a, b, c)$, and*

$$\begin{aligned}\tau' + \varepsilon_1 &= (a, -ia, i, 1), & -\tau' + \varepsilon_1 &= (a, ia, i, 1) \\ \tau' + \varepsilon_2 &= (b, 1, -ib, i), & -\tau' + \varepsilon_2 &= (b, 1, ib, i), \\ \tau' + \varepsilon_3 &= (c, i, 1, -ic), & -\tau' + \varepsilon_3 &= (c, i, 1, ic).\end{aligned}$$

Proof. An explicit formula for the translation automorphism $q \mapsto q + \tau$ is given in [34, Cor. 2.8]. Applying it to τ' gives $\tau' + \tau = (-abc, a, b, c)$. However, by [9, Eq. (7-2) and Prop. 7.4(3)], if $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in E$, then $-(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (-\lambda_0, \lambda_1, \lambda_2, \lambda_3)$. Hence $(-abc, a, b, c) = -\tau'$ and $2\tau' = -\tau$. The formulas for $\pm\tau' + \varepsilon_j$ come from Corollary 2.12. ■

The quadrics defined by the equations in (2-2) can now be labelled according to the system described in §2.2. These equations are similar to the central elements of S in Proposition 2.8.

Proposition 2.15. *The quadrics $Q(\tau + \xi)$, $\xi \in E[2]$, are*

$$\begin{aligned}Q(\tau) &= \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\}, \\ Q(\tau + \xi_1) &= \{x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2 = 0\}, \\ Q(\tau + \xi_2) &= \{x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2 = 0\}, \\ Q(\tau + \xi_3) &= \{x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2 = 0\}.\end{aligned}$$

Proof. The line $x_0 + ix_1 = x_2 - ix_3 = 0$ lies on the quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ and passes through the points $-\tau' + \varepsilon_1 = (a, ia, i, 1)$ and $-\tau' - \varepsilon_1 = (-\tau' + \varepsilon_1) - \xi_1 = (a, ia, -i, -1)$. Thus, the secant line through $-\tau' + \varepsilon_1$ and $-\tau' - \varepsilon_1$ lies on the quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. Hence that quadric is $Q(-\tau' + \varepsilon_1 - \tau' - \varepsilon_1) = Q(\tau)$.

We will do one more case and leave the other two cases to the reader.

The line $x_0 - icx_1 = icx_2 + x_3 = 0$ lies on the quadric $x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2 = 0$ and passes through the points $(ic, 1, -i, -c)$ and $(ic, 1, i, c)$. It follows from the various formulas above that $(ic, 1, -i, -c) = (-\tau' + \varepsilon_2) + \varepsilon_1$ and $(ic, 1, i, c) = (-\tau' + \varepsilon_2) - \varepsilon_1$. Since $\tau + \xi_2$ is the sum of $(-\tau' + \varepsilon_2) + \varepsilon_1$ and $(-\tau' + \varepsilon_2) - \varepsilon_1$ the line through the latter two points lies on $Q(\tau + \xi_2)$. Therefore $Q(\tau + \xi_2)$ is the quadric $x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2 = 0$. ■

2.9. Some 2-dimensional simple S -modules $V(\tau + \xi)$, $\xi \in E[2]$. This section concerns ungraded S -modules.

The finite dimensional simple S -modules were classified in [31] and [35] when $k = \mathbb{C}$ and τ has infinite order: for each integer $n \geq 0$, there are four 1-parameter families of simple S -modules of dimension $n + 1$ which were labelled $V(n\tau + \xi)^\lambda$, $\xi \in E[2]$, $\lambda \in k^\times$, in [35]. In [31], Sklyanin defined them by having the x_j 's act as difference operators on certain spaces of theta functions.

Proposition 2.16 describes four 2-dimensional simple modules that exist for all k and all τ whose order is not 2 or 4. The four 1-parameter families of 2-dimensional simple S -modules can be obtained from these four modules in the following way: if V is a left S -module and $\lambda \in k^\times$, let V^λ be the vector space V with a new action of S in which each $x \in S_1$ acts on V^λ as λx acts on V .

2.9.1. Quaternions. Define

$$(2-7) \quad q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $q_1^2 = q_2^2 = q_3^2 = -1$ and if (i, j, k) is a cyclic permutation of $(1, 2, 3)$, then $q_i q_j = q_k$ and $q_i q_j + q_j q_i = 0$.

Proposition 2.16. *For $j = 0, 1, 2, 3$, let $\rho_j : S \rightarrow M_2(k)$ be the homomorphism with $\rho_j(x_i)$ equal to the entry in row ρ_j and column x_i of Table 3. The ρ_j 's are structure maps for four pairwise non-isomorphic 2-dimensional simple S -modules, $V(\tau + \xi_j)$.*

		x_0	x_1	x_2	x_3
$V(\tau)$	ρ_0	1	q_1	q_2	q_3
$V(\tau + \xi_1)$	ρ_1	bcq_1	$-i$	$-ibq_3$	$-cq_2$
$V(\tau + \xi_2)$	ρ_2	acq_2	$-aq_3$	$-i$	$-icq_1$
$V(\tau + \xi_3)$	ρ_3	abq_3	$-iaq_2$	$-bq_1$	$-i$

TABLE 3. 2-dimensional simple S -modules.

If $1 \leq j \leq 3$, then $V(\tau + \xi_j) \cong \phi_j^* V(\tau)$ where ϕ_j is the automorphism in [Proposition 2.7](#).

Proof. First, $V(\tau)$ is an S -module because the elements in the row labelled $V(\tau)$ satisfy the relations

$$\begin{aligned} [x_0, x_1] &= 0 = \{x_2, x_3\} & \{x_0, x_1\} &= 2x_1 = [x_2, x_3] \\ [x_0, x_2] &= 0 = \{x_3, x_1\} & \{x_0, x_2\} &= 2x_2 = [x_3, x_1] \\ [x_0, x_3] &= 0 = \{x_1, x_2\} & \{x_0, x_3\} &= 2x_3 = [x_1, x_2] \end{aligned}$$

and therefore satisfy the six defining relations for S . Since $M_2(k)$ is generated as an algebra by $\{1, q_1, q_2, q_3\}$, $V(\tau)$ is a simple S -module.

The other three modules are obtained from $V(\tau)$ by applying the autoequivalences ϕ_j^* of $\text{Mod}(S)$ induced by the k -algebra automorphisms ϕ_j in [Proposition 2.7](#). Explicitly, $\phi_j^*(V(\tau))$ is obtained by having each x_i act on k^2 by $\phi_j(x_i)$. Thus, applying ϕ_j to the elements in row $V(\tau)$ gives the elements in the row $V(\tau + \xi_j)$. Of course, $V(\tau + \xi_j)$ is simple because $V(\tau)$ is.

Let $j \in \{1, 2, 3\}$. The four modules are pairwise non-isomorphic because $x_j + i$ annihilates $V(\tau + \xi_j)$ but none of the other three modules, and $x_0 - 1$ annihilates $V(\tau)$ but not $V(\tau + \xi_j)$. ■

It is easy to see that $V(\tau + \xi_j)$ is annihilated by the central element

$$\begin{cases} \Omega + 4 & \text{if } j = 0, \\ x_0^2 + \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2 + 4\beta\gamma & \text{if } j = 1, \\ x_0^2 + \gamma x_1^2 + \alpha\gamma x_2^2 - \alpha x_3^2 + 4\alpha\gamma & \text{if } j = 2, \\ x_0^2 - \beta x_1^2 + \alpha x_2^2 + \alpha\beta x_3^2 + 4\alpha\beta & \text{if } j = 3. \end{cases}$$

The next proof uses the duality between lines in $\mathbb{P}(S_1)$ and lines in $\mathbb{P}(S_1^*)$ that is induced by the duality $X \longleftrightarrow X^\perp$ between 2-dimensional subspaces of S_1 and 2-dimensional subspaces of S_1^* .

Proposition 2.17. *Let $\xi \in E[2]$ and $p, q \in E$. If $p + q = \tau + \xi$, then*

- (1) *there is a surjective homomorphism $M_{p,q} \rightarrow V(\tau + \xi)$ in $\text{Mod}(S)$ and*
- (2) *$\text{Hom}_S(M_{p,q}, V(\tau + \xi)) \cong k$.*

Proof. (1) Consider the case $\xi = o$. The element $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V(\tau)$ is annihilated by $x_0 + ix_1$ and $x_2 - ix_3$. In the proof of [Proposition 2.15](#), we observed that the line $x_0 + ix_1 = x_2 - ix_3 = 0$ is the secant line through the points $-\tau' + \varepsilon_1$ and $-\tau' - \varepsilon_1$. Hence, there is a surjective homomorphism

$$M_{-\tau' + \varepsilon_1, -\tau' - \varepsilon_1} = \frac{S}{S(x_0 + ix_1) + S(x_2 - ix_3)} \twoheadrightarrow V(\tau).$$

The action of S on $V(\tau)$ restricts to a linear isomorphism $\rho : S_1 \rightarrow \text{END}_k V(\tau)$. We will also write ρ for the induced isomorphism $\mathbb{P}(S_1) \rightarrow \mathbb{P}(\text{END}_k V(\tau))$. Let $Q \subseteq \mathbb{P}(\text{END}_k V(\tau))$ be the quadric where the determinant vanishes. Let \mathbf{L} denote the set of lines on Q that are the images of the simple left ideals in $\text{END}_k V(\tau)$. The lines in \mathbf{L} provide a ruling on Q . The preimage $\rho^{-1}(Q)$ of Q in $\mathbb{P}(S_1)$ is also a smooth quadric and $\rho^{-1}(\mathbf{L}) = \{\rho^{-1}(\ell) \mid \ell \in \mathbf{L}\}$ is a ruling on $\rho^{-1}(Q)$.

If $x = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, then $\det(\rho(x)) = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ so $\rho^{-1}(Q) = \{\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \mid \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0\}$. It is easy to verify the following fact: if $\ell \in \rho^{-1}(\mathbf{L})$, then the line $\ell^\perp := \{\text{the points where } \ell \text{ vanishes}\} \subseteq \mathbb{P}(S_1^*)$ lies on the quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$, i.e., on $\Omega(\tau)$. Therefore $\{\ell^\perp \mid \ell \in \rho^{-1}(\mathbf{L})\}$ is a ruling on $\Omega(\tau)$. Since the secant line through $-\tau' + \varepsilon_1$ and $-\tau' - \varepsilon_1$ belongs to $\{\ell^\perp \mid \ell \in \rho^{-1}(\mathbf{L})\}$ it follows that

$$\{\ell^\perp \mid \rho(\ell) \in \mathbf{L}\} = \{\overline{pq} \mid p, q \in E \text{ and } p + q = \tau\}.$$

This equality says that if $p + q = \tau$, then $S(\overline{pq})^\perp$ annihilates a non-zero element in $V(\tau)$ so there is a surjective map

$$M_{p,q} = \frac{S}{S(\overline{pq})^\perp} \twoheadrightarrow V(\tau).$$

This completes the proof of (1) when $\xi = o$.

The other cases are obtained by applying the auto-equivalences ϕ_j^* , $j = 1, 2, 3$, to the case $\xi = o$. There is a surjective homomorphism $\phi_j^* M_{p,q} \rightarrow \phi_j^* V(\tau) = V(\tau + \xi_j)$. However, because ϕ_j is a scalar multiple of ε_j , $\phi_j^* M_{p,q} \cong \varepsilon_j^* M_{p,q} \cong M_{p+\varepsilon_j, q+\varepsilon_j}$.² In conclusion, if $p, q \in E$ are such that $p + q = \tau + \xi_j$, then there is a surjective homomorphism $M_{p,q} \rightarrow V(\tau + \xi_j)$.

(2) Arguing as in the previous paragraph, it suffices to prove (2) when $\xi = o$. Since $M_{p,q}$ is a cyclic module and $\dim_k(V(\tau)) = 2$, the dimension of $\text{Hom}_S(M_{p,q}, V(\tau + \xi))$ is at most two. However, if it were two, then $(\overline{pq})^\perp$ would annihilate $V(\tau + \xi)$. If $(p') + (q')$ is another divisor on E such that $p' + q' = \tau$, then the lines \overline{pq} and $\overline{p'q'}$ belong to the same ruling on the smooth quadric $Q(\tau)$ so are disjoint. Hence $(\overline{pq})^\perp + (\overline{p'q'})^\perp = S_1$. By (1) applied to p' and q' in place of p and q there is a non-zero element in $V(\tau)$ that is annihilated by $(\overline{p'q'})^\perp$ and hence by S_1 . This can not happen because $V(\tau)$ is a simple module of dimension 2. We conclude that the dimension of $\text{Hom}_S(M_{p,q}, V(\tau + \xi))$ is one. ■

Corollary 2.18. *If $\xi \in E[2]$, then $V(\tau + \xi)$ is annihilated by $\Omega(\tau + \xi)$.*

2.10. Multiplicity-two fat point modules in $\text{Gr}(S)$. Let $k[t]$ be the polynomial ring in one variable. As in [35, Rmk. 2, p.79], for each $\xi \in E[2]$ we make

$$\tilde{V}(\tau + \xi) := V(\tau + \xi) \otimes k[t]$$

a graded left S -module by declaring that $\deg(v \otimes t^m) = m$ and having $a \in A_n$ act on $v \otimes t^m$ as $a(v \otimes t^m) = (av) \otimes t^{m+n}$.

Clearly, for all $n \geq 0$, the graded A -modules $\tilde{V}(\tau + \xi)(n)_{\geq 0}$ and $\tilde{V}(\tau + \xi)$ are isomorphic.

When $k = \mathbb{C}$ and τ has infinite order, the next result is a special case of results in [35, §4].

Proposition 2.19. *Let $\xi \in E[2]$ and let $p, q \in E$. If $p + q = \tau + \xi$, then*

- (1) $\text{Hom}_{\text{Gr}(S)}(M_{p,q}, \tilde{V}(\tau + \xi)) \cong k$;
- (2) every non-zero $f \in \text{Hom}_{\text{Gr}(S)}(M_{p,q}, \tilde{V}(\tau + \xi)) \cong k$ is surjective in degrees ≥ 1 ;
- (3) $\tilde{V}(\tau + \xi)$ is 1-critical of multiplicity 2;
- (4) there is an exact sequence $0 \rightarrow M_{p-2\tau, q-2\tau}(-2) \rightarrow M_{p,q} \xrightarrow{f} \text{im}(f) \rightarrow 0$.

Proof. (1) Let $\tilde{f} : M_{p,q} \rightarrow V(\tau + \xi)$ be a surjective S -module homomorphism as in Proposition 2.17. For $m \in (M_{p,q})_n$ define $f(m) := \tilde{f}(m) \otimes t^n$. It is easy to see that f is a morphism in $\text{Gr}(S)$ (cf., [35, Rmk. 2, p.79]). The argument used to prove Proposition 2.17(2) can be adapted to show that the dimension of $\text{Hom}_{\text{Gr}(S)}(M_{p,q}, \tilde{V}(\tau + \xi))$ is one.

(2) Since the structure map $S_1 \rightarrow \text{End}_k V(\tau + \xi)$ is surjective (see Table 3), $S_1(v \otimes 1) = V(\tau + \xi) \otimes t$ for all non-zero v in $V(\tau + \xi)$. Therefore $f((M_{p,q})_1) = \tilde{V}(\tau + \xi)_1$. The result follows.

²This last isomorphism is discussed in the proof of Corollary 2.10.

(3) Certainly, $\tilde{V}(\tau + \xi)$ has multiplicity 2. If v is a non-zero element in $V(\tau + \xi)$, then $S_1 v = V(\tau + \xi)$. Hence $S_1(v \otimes t^n) = \tilde{V}(\tau + \xi)_{n+1}$ for all $n \geq 0$. It follows that every proper graded quotient module of $\tilde{V}(\tau + \xi)$ has finite dimension. Hence $\tilde{V}(\tau + \xi)$ is 1-critical.

(4) This is a special case of [35, Prop. 4.4]. ■

2.10.1. *Some functors.* Let R be a graded k -algebra.

We will now define a functor $G : \text{Mod}(R) \rightarrow \text{Gr}(R)$. Let $k[t]$ be the polynomial ring in one variable. If V is a left R -module we make $G(V) := V \otimes k[t]$ a graded R -module by declaring that $(GV)_n = V \otimes t^n$ and making $r \in R_d$ act on $v \otimes t^n$ by $r(v \otimes t^n) = (rv) \otimes t^{n+d}$. If $h : V \rightarrow V'$ is a homomorphism of left R -modules we define $G(h) : G(V) \rightarrow G(V')$ by $G(h)(v \otimes t^n) = h(v) \otimes t^n$.

If $\lambda \in k^\times$, let $\psi_\lambda \in \text{Aut}_{\text{gr}}(R)$ be the graded k -algebra automorphism defined by $\psi_\lambda(r) = \lambda^n r$ for $r \in R_n$. The auto-equivalence $\psi_\lambda^* : \text{Gr}(R) \rightarrow \text{Gr}(R)$ is defined as follows:

- (1) if $M \in \text{Gr}(R)$ the underlying graded vector space for $\psi_\lambda^* M$ is M ;
- (2) if $m \in M$ we write m_λ for m viewed as an element in $\psi_\lambda^* M$;
- (3) the action of $r \in R_d$ on $m_\lambda \in \psi_\lambda^* M$ is $rm_\lambda = \lambda^d (rm)_\lambda$;
- (4) if $h : M \rightarrow N$ is a morphism in $\text{Gr}(R)$, then $\psi_\lambda^*(h) = h$.

Lemma 2.20. *Let R be a graded k -algebra, $\lambda \in k^\times$, $V \in \text{Mod}(R)$, and $M \in \text{Gr}(R)$.*

- (1) *There is an isomorphism $f : M \rightarrow \psi_\lambda^* M$ in $\text{Gr}(R)$ given by $f(m) = \lambda^n m$ for $m \in M_n$.*
- (2) *There are isomorphisms $\psi_\lambda^* GV \xleftarrow{f} GV \xrightarrow{\theta} G\psi_\lambda^* V$ in $\text{Gr}(R)$ given by $f(v \otimes t^n) = \lambda^n (v \otimes t^n)_\lambda$ and $\theta(v \otimes t^n) = \lambda^n v_\lambda \otimes t^n$.*
- (3) *If $\phi \in \text{Aut}_{\text{gr}}(R)$, then $G(\phi^* V) \cong \phi^*(G(V))$ in $\text{Gr}(R)$.*

Proof. (1) Certainly, f is an isomorphism of graded vector spaces. Let $r \in R_j$ and $m \in M_n$. Then $r(f(m)_\lambda) = r(\lambda^n m_\lambda) = \lambda^j \lambda^n (rm)_\lambda = \lambda^{n+j} (rm)_\lambda = f(rm)$. Hence f is an R -module homomorphism and therefore an isomorphism of graded R -modules.

(2) Certainly, θ is an isomorphism of graded vector spaces. If $r \in R_d$, then $\theta(r(v \otimes t^n)) = \theta((rv) \otimes t^{n+d}) = \lambda^{n+d} (rv)_\lambda \otimes t^{n+d}$ and $r\theta(v \otimes t^n) = r(\lambda^n v_\lambda \otimes t^n) = \lambda^n r(v_\lambda \otimes t^n) = \lambda^n \lambda^d (rv)_\lambda \otimes t^{n+d}$. Hence $\theta(r(v \otimes t^n)) = r\theta(v \otimes t^n)$; i.e., θ is an R -module homomorphism and hence an isomorphism in $\text{Gr}(R)$. The function f in part (2) is the isomorphism f in part (1) applied to $M = GV$.

(3) By definition, $\phi^* V$ is equal to V as a vector space and we write v_ϕ when an element $v \in V$ is considered as an element of $\phi^* V$. Thus, $rv_\phi = (\phi(r)v)_\phi$ for all $r \in R$ and $v \in V$.

The function $g : G(\phi^* V) \rightarrow \phi^*(G(V))$ given by $g(v_\phi \otimes t^n) = (v \otimes t^n)_\phi$ is an isomorphism of graded vector spaces. It is also a homomorphism of R -modules because if $r \in R_d$, then

$$g(r(v_\phi \otimes t^n)) = g(rv_\phi \otimes t^{n+d}) = g((\phi(r)v)_\phi \otimes t^{n+d}) = (\phi(r)v \otimes t^{n+d})_\phi$$

and

$$rg(v_\phi \otimes t^n) = r(v \otimes t^n)_\phi = (\phi(r)(v \otimes t^n))_\phi = (\phi(r)v \otimes t^{n+d})_\phi.$$

Hence g is an isomorphism in $\text{Gr}(R)$. ■

Proposition 2.21.

- (1) $\gamma^* V(\tau) \cong V(\tau)$ for all $\gamma \in \Gamma = \langle \varepsilon_1^2, \varepsilon_2^2 \rangle \subseteq H_4$.
- (2) $\gamma^* \tilde{V}(\tau + \xi_j) \cong \tilde{V}(\tau + \xi_j)$ for all $\gamma \in \Gamma \times Z = \langle \varepsilon_1^2, \varepsilon_2^2, \delta \rangle \subseteq H_4$ and all $j \in \{0, 1, 2, 3\}$.
- (3) $\phi_j^* \tilde{V}(\tau) \cong \tilde{V}(\tau + \xi_j)$ for $j \in \{0, 1, 2, 3\}$.

Proof. (1) Let $\rho : S_1 \rightarrow M_2(k)$ be the restriction to S_1 of the structure map for $V(\tau)$; i.e., $\rho(x_i)$ is the entry in row $V(\tau)$ and column x_i of Table 3.

We write $\gamma_0 = 1, \gamma_1, \gamma_2, \gamma_3$ for the elements of Γ as in Table 1.

We will show that the function $\theta : k^2 \rightarrow k^2$, $\theta(v) = q_j v$, is an isomorphism of left S -modules $V(\tau) \rightarrow \gamma_j^* V(\tau)$ by showing that $\theta(\rho(x)v) = \rho_{\gamma_j}(x)\theta(v)$ for all $x \in S_1$ and $v \in k^2$. To do this it

suffices to show that $q_j \rho(x_i) = \rho \gamma_j(x_i) q_j$ for $i = 0, 1, 2, 3$. This is true because

$$\rho \gamma_j(x_i) q_j = \begin{cases} \rho(x_i) q_j = q_i q_j = q_j q_i = q_j \rho(x_i) & \text{if } j \in \{0, i\}, \\ \rho(-x_i) q_j = -q_i q_j = q_j q_i = q_j \rho(x_i) & \text{if } j \notin \{0, i\}. \end{cases}$$

(2) We first prove that $\gamma^* \tilde{V}(\tau) \cong \tilde{V}(\tau)$. The elements in $Z = \langle \delta \rangle \subseteq H_4$ act on S_1 as scalar multiplication. Thus, if $\gamma \in \Gamma \times Z$, then $\gamma^* = \gamma_j^* \psi_\lambda^*$ for some $\lambda \in \{\pm i, \pm 1\}$ and some $j \in \{0, 1, 2, 3\}$. By Lemma 2.20, $\gamma_j^* \psi_\lambda^* \tilde{V}(\tau) \cong G(\gamma_j^* V(\tau)) \cong G(V(\tau)) = \tilde{V}(\tau)$.

(3) By definition, $V(\tau + \xi_j) = \phi_j^* V(\tau)$ so, by Lemma 2.20(3), $\phi_j^* \tilde{V}(\tau) = \phi_j^* G(V(\tau)) \cong G \phi_j^* V(\tau) = G(V(\tau + \xi_j)) = \tilde{V}(\tau + \xi_j)$.

Now we can finish the proof of (2). Since $\phi_j \gamma = \psi_\lambda \gamma \phi_j$ for some $\lambda \in \{\pm i, \pm 1\}$,

$$\gamma^* \tilde{V}(\tau + \xi_j) = \gamma^* \phi_j^* \tilde{V}(\tau) = (\phi_j \gamma)^* \tilde{V}(\tau) = \phi_j^* \gamma^* \psi_\lambda^* \tilde{V}(\tau) \cong \phi_j^* \gamma^* \tilde{V}(\tau) \cong \phi_j^* \tilde{V}(\tau) \cong \tilde{V}(\tau + \xi_j)$$

where, again, we used Lemma 2.20(3). ■

3. PRELIMINARIES CONCERNING $A = A(E, \tau)$

3.1. The definition of A . We make Γ act as k -algebra automorphisms of $M_2(k)$ by $\gamma_j(a) := q_j a q_j^{-1}$ and define

$$A := (S \otimes M_2(k))^\Gamma$$

where Γ acts diagonally as graded k -algebra automorphisms of $S \otimes M_2(k)$. The elements

$$y_0 := x_0 \otimes 1, \quad y_1 := x_1 \otimes q_1, \quad y_2 := x_2 \otimes q_2, \quad y_3 := x_3 \otimes q_3,$$

are a basis for A_1 and A is isomorphic to the free algebra $k\langle y_0, y_1, y_2, y_3 \rangle$ modulo the relations

$$(3-1) \quad y_0 y_i - y_i y_0 = \alpha_i (y_j y_k - y_k y_j) \quad \text{and} \quad y_0 y_i + y_i y_0 = y_j y_k + y_k y_j,$$

See [9, §6] for more information.

3.1.1. Γ -equivariant S' -modules. We define $S' := S \otimes M_2(k)$ and write $\text{Gr}(S')^\Gamma$ for the category of Γ -equivariant graded left S' -modules. By [9, Prop. 3.9], there is an equivalence of categories $\text{Gr}(S')^\Gamma \cong \text{Gr}(A)$ given by sending a Γ -equivariant S' -module M to M^Γ , its Γ -invariant subspace.

We usually view a Γ -equivariant S' -module as a left S' -module M endowed with a left action $\Gamma \times M \rightarrow M$, $(\gamma, m) \mapsto m^\gamma$, such that $(xm)^\gamma = \gamma(x)m^\gamma$ for all $x \in S'$, $m \in M$, and $\gamma \in \Gamma$. We often describe this by saying that the actions of Γ and S' on M are **compatible**.

3.1.2. Twisting S by a 2-cocycle. Let $\hat{\Gamma} = \{1, \chi_1, \chi_2, \chi_3\}$ denote the character group of Γ . Since the characteristic of k does not divide $|\Gamma|$, there is a $\hat{\Gamma}$ -grading on S ,

$$S = \bigoplus_{\chi \in \hat{\Gamma}} S_\chi = \bigoplus_{\chi \in \hat{\Gamma}} \{a \in S \mid \gamma(a) = \chi(\gamma)a \text{ for all } \gamma \in \Gamma\}.$$

Let $\mu : \hat{\Gamma} \times \hat{\Gamma} \rightarrow k^\times$ be the 2-cocycle $\mu(\chi_1^p \chi_2^q, \chi_1^r \chi_2^s) := (-1)^{ps}$. In his Ph.D. thesis [15, Prop. 3.1.16], Davies showed that S with the new multiplication $w * w' := \mu(\chi, \chi') w w'$ if $w \in S_\chi$ and $w' \in S_{\chi'}$ is isomorphic to A .

In §6.2, we give a slightly different interpretation of this construction of A by starting with the homomorphism $H_4 \rightarrow \text{Aut}(S)$ and viewing S as an algebra object in the category of comodules over the ring $k(H_4)$ of k -valued functions on H_4 .

3.2. Our convention for identifying A_1 and S_1 . From the perspective in §3.1.2, S and A have the same underlying \mathbb{Z} -graded vector space but different algebra structures.

In terms of the generators x_i and y_i we adopt the convention that S_1 and A_1 are identified via

$$x_0 = y_0, \quad x_1 = iy_1, \quad x_2 = iy_2, \quad x_3 = y_3.$$

We use this convention whenever we refer to S and A as being supported on the same graded vector space. Later on, we will use the fact that this is the identification used in [9, Prop. 10.10].

3.2.1. By [9, Prop. 6.2], Γ also acts as automorphisms of A . In fact, [9, Prop. 6.2] shows that if one identifies S_1 and A_1 according to the above convention, then the action of Γ on $S_1 = A_1$ extends to an action as k -algebra automorphisms of both S and A . We will sometimes use this convention to obtain an action of H_4 on $S_1 = A_1$; this action of H_4 extends to an action of H_4 as k -algebra automorphisms of S but not of A .

3.3. The centers of S and A . Let Ω_j , $j = 0, 1, 2, 3$, be the central elements of S defined in Proposition 2.2. The only central elements in S that play a role in this paper are those in the subalgebra $k[\Omega_0, \Omega_1] = k[\Omega_0, \Omega_1, \Omega_2, \Omega_3]$. Each Ω_j is fixed by Γ so $k[\Omega_0, \Omega_1] \otimes 1 \subseteq (S \otimes M_2(k))^\Gamma = A$. We will identify $k[\Omega_0, \Omega_1]$ with this subalgebra of A . Since $k[\Omega_0, \Omega_1] \otimes 1$ is in the center of $S \otimes M_2(k)$ this copy of $k[\Omega_0, \Omega_1]$ belongs to the center of A .

We will take advantage of this fact by using the notation $\Omega(z)$, $z \in E$, for the central element $\Omega(z) \otimes 1$ in A . Each $\Omega(z)$ is a linear combination of x_0^2 , x_1^2 , x_2^2 , and x_3^2 . As an element in A , $\Omega(z)$, which is really $\Omega(z) \otimes 1$, is a linear combination of y_0^2 , y_1^2 , y_2^2 , and y_3^2 . Since $y_j = x_j \otimes q_j$,

$$y_j^2 = \begin{cases} x_0^2 \otimes 1 & \text{if } j = 0, \\ -x_j^2 \otimes 1 & \text{if } j \neq 0. \end{cases}$$

Thus, when considered as elements of A , the central elements in (2-4) are

$$(3-2) \quad \begin{cases} \Omega &= -y_0^2 - y_1^2 - y_2^2 - y_3^2, \\ \Theta_1 &:= y_0^2 - \beta\gamma y_1^2 + \gamma y_2^2 - \beta y_3^2, \\ \Theta_2 &:= y_0^2 - \gamma y_1^2 - \alpha\gamma y_2^2 + \alpha y_3^2, \\ \Theta_3 &:= y_0^2 + \beta y_1^2 - \alpha y_2^2 - \alpha\beta y_3^2. \end{cases}$$

3.4. The ring \tilde{B} . Let \tilde{B} be the quotient of A by the ideal generated by all $\Omega(z)$. By [9, §8], there is an equivalence of categories $\mathbf{QGr}(\tilde{B}) \equiv \mathbf{Qcoh}(E/E[2])$. Under this equivalence the skyscraper sheaf $\mathcal{O}_{p+E[2]}$ at the point $p + E[2]$ corresponds to the image of $M_p \otimes k^2$ where M_p is the point module in $\mathbf{Gr}(S)$ corresponding to p and A acts on $M_p \otimes k^2$ by virtue of the fact that S is a subalgebra of $S \otimes M_2(k)$. If $\omega \in E[2]$, then $M_p \otimes k^2$ and $M_{p+\omega} \otimes k^2$ are isomorphic in $\mathbf{Gr}(A)$.

We call the A -module $M_p \otimes k^2$ a fat point of multiplicity two; “fat” because its Hilbert series is $2(1-t)^{-1}$ and “point” because it is irreducible as an object in $\mathbf{QGr}(A)$.

3.5. Elliptic line modules. Let $p \in E$ and write $x = p + E[2] \in E/E[2]$. If ξ , ξ' , and ξ'' are the three 2-torsion points, we define the S' -module

$$(3-3) \quad M_{x,\xi} := (M_{p,p+\xi} \oplus M_{p+\xi',p+\xi'+\xi}) \otimes k^2.$$

Proposition 3.1. [9, §10] *If ξ and ω are 2-torsion points and $x, y \in E/E[2]$, then $M_{x,\xi} \cong M_{y,\omega}$ as S' -modules if and only if $(x, \xi) = (y, \omega)$.*

In [9, §10], we showed there are exactly two Γ -equivariant structures on each $M_{x,\xi}$ and then, by taking the Γ -invariant subspace of the equivariant structures on the $M_{x,\xi}$ ’s, we constructed line modules for A parametrized by $(E/\langle \xi \rangle) \sqcup (E/\langle \xi' \rangle) \sqcup (E/\langle \xi'' \rangle)$. We call these elliptic line modules.

3.6. Other remarks, conventions, and notation.

3.6.1. Symmetries in A . Calculations in A can often be carried out more efficiently by exploiting the symmetries involving the ordered triples $(\alpha_1, \alpha_2, \alpha_3)$ and (y_1, y_2, y_3) . To formalize this idea let α_1, α_2 , and α_3 , denote central indeterminates and define the algebra $\widehat{S} := k[\alpha_1, \alpha_2, \alpha_3][y_0, y_1, y_2, y_3]$ with relations (3-1). There is a k -algebra automorphism $\phi : \widehat{S} \rightarrow \widehat{S}$ given by $\phi(y_0) = y_0$, and $\phi(\alpha_j) = \alpha_{j+1}$ and $\phi(y_j) = y_{j+1}$ for $j \in \{1, 2, 3\} = \mathbb{Z}/(3)$. Most explicit calculations in A can be interpreted as calculations in \widehat{S} followed by the obvious specialization $\widehat{S} \rightarrow A$.

For example, if one wants to show that the elements $\Delta_i := y_0^2 - \alpha_j \alpha_k y_i^2 + \alpha_k y_j^2 - \alpha_j y_k^2$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, are central in A it suffices to show that the corresponding element $\widehat{\Delta}_1 \in \widehat{S}$ is central in \widehat{S} , then apply ϕ and ϕ^2 to $\widehat{\Delta}_1$ to see that $\widehat{\Delta}_2$ and $\widehat{\Delta}_3$ are central in \widehat{S} , then specialize to A to see that Δ_1, Δ_2 , and Δ_3 , are central in A .

3.6.2. Comodules. If H is a k -coalgebra we write \mathcal{M}^H for the category of right H -comodules. If $\dim_k(H) < \infty$, then \mathcal{M}^H is equivalent to the category $\text{Mod}(H^*)$ of left H^* -modules.

3.6.3. Invariant modules. Let Γ be a group acting as automorphisms of a ring R . If $\gamma \in \Gamma$ we denote by γ^* the following automorphism of $\text{Mod}(R)$: if $f : M \rightarrow N$ is a morphism in $\text{Mod}(R)$, γ^*M is M as an abelian group with a new action of R , $r * m = \gamma(r)m$, and $\gamma^*(f) = f$. If $\gamma^*M \cong M$ for all $\gamma \in \Gamma$ we say that M is invariant under the action of Γ .

3.6.4. Hilbert series, multiplicity, and criticality. The Hilbert series of a finitely generated graded module M over S , or A , or $A \otimes M_2(k)$, is the formal series $H(M; t) = \sum_{n \in \mathbb{Z}} \dim_k(M_n) t^n$. Because these rings have finite global dimension and Hilbert series $(1 - t)^{-4}$ for S and A , and $4(1 - t)^{-4}$ for $A \otimes M_2(k)$, $H(M; t) = f(t)(1 - t)^{-d}$ for some integer d and some $f(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $f(1) \neq 0$. The integer d is the Gelfand-Kirillov dimension, or GK-dimension of M . We call $f(1)$ the multiplicity of M . If $\text{GKdim}(M) = d$ and $\text{GKdim}(M/N) < d$ for all non-zero $N \subseteq M$ we say that M is d -critical. If M is 1-critical, then M becomes a simple object in the quotient category QGr .

4. POINT MODULES FOR A

The point modules for A were classified in [9, §9] by a rather unilluminating calculation. In this section we show they can be obtained in a more systematic and meaningful way. Let $j \in \{0, 1, 2, 3\}$ and write $F = \widehat{V}(\tau + \xi_j)$. We will show there are four Γ -equivariant structures on the S' -module $F \otimes k^2$, and that the four point modules $(F \otimes k^2)^\Gamma$ for A are the four point modules in \mathfrak{P}_j below.

4.1. Classification of point modules for A . If M is a point module for A , then $M \cong A/Ap^\perp$ for a unique $p \in \mathbb{P}(A_1^*)$. By [9, §9], A has 20 point modules up to isomorphism and, with respect to the coordinate functions (y_0, y_1, y_2, y_3) on $\mathbb{P}(A_1^*)$, the corresponding points are those in Table 4.

\mathfrak{P}_∞	\mathfrak{P}_0	\mathfrak{P}_1	\mathfrak{P}_2	\mathfrak{P}_3	Γ
$(1, 0, 0, 0)$	$(1, 1, 1, 1)$	$(bc, -i, -ib, -c)$	$(ac, -a, -i, -ic)$	$(ab, -ia, -b, -i)$	
$(0, 1, 0, 0)$	$(1, 1, -1, -1)$	$(bc, -i, ib, c)$	$(ac, -a, i, ic)$	$(ab, -ia, b, i)$	γ_1
$(0, 0, 1, 0)$	$(1, -1, 1, -1)$	$(bc, i, -ib, c)$	$(ac, a, -i, ic)$	$(ab, ia, -b, i)$	γ_2
$(0, 0, 0, 1)$	$(1, -1, -1, 1)$	$(bc, i, ib, -c)$	$(ac, a, i, -ic)$	$(ab, ia, b, -i)$	γ_3

TABLE 4. The points in \mathfrak{P} .

For example, if $p = (bc, -i, -ib, -c)$ the corresponding point module is

$$M_p = \frac{A}{Ap^\perp} = \frac{A}{A(ix_0 + bcx_1) + A(ix_0 + cx_2) + A(x_0 + bx_3)}.$$

Each point in \mathfrak{P}_∞ is fixed by Γ . If $j \neq \infty$, then \mathfrak{P}_j is a Γ -orbit and the points in the column labelled \mathfrak{P}_j are, in descending order, p , $\gamma_1(p)$, $\gamma_2(p)$, and $\gamma_3(p)$.

The involution $\theta : \mathfrak{P} \rightarrow \mathfrak{P}$ is defined by

$$(4-1) \quad \theta(p) := \begin{cases} p & \text{if } p \in \mathfrak{P}_\infty \cup \mathfrak{P}_0 \\ \gamma_i(p) & \text{if } p \in \mathfrak{P}_i, i = 1, 2, 3. \end{cases}$$

If M_p is the point module corresponding to $p \in \mathfrak{P}$, then $M_p(1)_{\geq 0} \cong M_{\theta(p)}$. In particular, if $p \in \mathfrak{P}_j$ and $j \neq \infty$, then $M_p(1)_{\geq 0} \cong M_{\gamma_j(p)} \cong \gamma_j^* M_p$.

We call point modules in \mathfrak{P}_∞ **special** and those in the other \mathfrak{P}_j 's **ordinary**.

Proposition 4.1. *Let $p \in \mathfrak{P} - \mathfrak{P}_\infty$. Then M_p is annihilated by*

$$\begin{cases} 4\Theta_1 + (1 - \beta)(1 + \gamma)\Omega & \text{if } p \in \mathfrak{P}_0, \\ 4\beta\gamma\Omega + (1 - \beta)(1 + \gamma)\Theta_1 & \text{if } p \in \mathfrak{P}_1, \\ 4\alpha\gamma\Omega + (1 - \gamma)(1 + \alpha)\Theta_2 & \text{if } p \in \mathfrak{P}_2, \\ 4\alpha\beta\Omega + (1 - \alpha)(1 + \beta)\Theta_3 & \text{if } p \in \mathfrak{P}_3, \end{cases}$$

where $\Theta_1, \Theta_2, \Theta_3$ are the central elements in (3-2).

Proof. Since the points in \mathfrak{P}_j form a single Γ -orbit and the elements y_0^2, \dots, y_3^2 are fixed by Γ , a linear combination of y_0^2, \dots, y_3^2 annihilates M_p for some $p \in \mathfrak{P}_j$ if and only if it annihilates M_p for all $p \in \mathfrak{P}_j$. Furthermore, if that linear combination is in the center of A it annihilates M_p if and only if it annihilates $(M_p)_0$.

We prove the result for $j \in \{0, 1\}$ and leave the reader to check the other two cases.

Let $M \in \text{Gr}(A)$ be the point module associated to $p = (1, 1, 1, 1) \in \mathfrak{P}_0$. Let e_0 be a basis for M_0 . There is an element $e_1 \in M_1$ such that $y_i e_0 = e_1$ for $i = 0, 1, 2, 3$. Since $\theta(1, 1, 1, 1) = (1, 1, 1, 1)$, there is an element $e_2 \in M_2$ such that $y_i e_1 = e_2$ for $i = 0, 1, 2, 3$. It follows that $y_i^2 e_0 = e_2$ for $i = 0, 1, 2, 3$. Hence $(y_0^2 + y_1^2 + y_2^2 + y_3^2)e_0 = 4e_2$. On the other hand, $\Theta_1 e_0 = (y_0^2 - \beta\gamma y_1^2 + \gamma y_2^2 - \beta y_3^2)e_0 = (1 - \beta)(1 + \gamma)e_2$. Therefore e_0 is annihilated by $4\Theta_1 - (1 - \beta)(1 + \gamma)(y_0^2 + y_1^2 + y_2^2 + y_3^2)$.

Let $M \in \text{Gr}(A)$ be the point module associated to $p = (bc, -i, -ib, -c) \in \mathfrak{P}_1$. Let e_0 be a basis for M_0 . There is an element $e_1 \in M_1$ such that $y_0 e_0 = bce_1$, $y_1 e_0 = -ie_1$, $y_2 e_0 = -ibe_1$, and $y_3 e_0 = -ce_1$. Since $\theta(p) = \gamma_1(p) = (bc, -i, ib, c)$, there is an element $e_2 \in M_2$ such that $y_0 e_1 = bce_2$, $y_1 e_1 = -ie_2$, $y_2 e_1 = ibe_2$, and $y_3 e_1 = ce_2$. Therefore $\Theta_1 e_0 = (y_0^2 - \beta\gamma y_1^2 + \gamma y_2^2 - \beta y_3^2)e_0 = 4\beta\gamma e_2$. On the other hand, $(y_0^2 + y_1^2 + y_2^2 + y_3^2)e_0 = (\gamma + 1)(\beta - 1)e_2$. Therefore e_0 is annihilated by $4\beta\gamma\Omega + (1 - \beta)(1 + \gamma)\Theta_1$. \blacksquare

4.2. The actions of H_4 on A_1 and \mathfrak{P} . We identify A_1 with S_1 according to the convention in §3.2. There are corresponding identifications of A_1^* and S_1^* , and of $\mathbb{P}(A_1^*)$ and $\mathbb{P}(S_1^*)$. If $p = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ is a point in S_1^* written with respect to the coordinate functions (x_0, x_1, x_2, x_3) , then $x_j(p) = \lambda_j$. Therefore $y_j(p) = \lambda_j$ if $j \in \{0, 3\}$ and $y_j(p) = -i\lambda_j$ if $j \in \{1, 2\}$. Thus, as a point in A_1^* , p has coordinates $(\lambda_0, -i\lambda_1, -i\lambda_2, \lambda_3)$ with respect to the coordinate functions (y_0, y_1, y_2, y_3) .

These identifications lead to actions of H_4 on A_1^* and $\mathbb{P}(A_1^*)$. When expressed in terms of the coordinate functions (y_0, y_1, y_2, y_3) , the action of H_4 on A_1^* is also given by the formulas in (2-6).

Proposition 4.2. *The action of H_4 on $\mathbb{P}(A_1^*)$ is such that:*

- (1) \mathfrak{P}_∞ is a single H_4 -orbit;
- (2) if $\{i, j, k\} = \{1, 2, 3\}$, then $\varepsilon_i(\mathfrak{P}_0) = \mathfrak{P}_i$, $\varepsilon_i(\mathfrak{P}_i) = \mathfrak{P}_0$, $\varepsilon_i(\mathfrak{P}_j) = \mathfrak{P}_k$, and $\varepsilon_i(\mathfrak{P}_k) = \mathfrak{P}_j$.

Proof. Case-by-case calculations using the formulas in (2-6) prove the result. For example,

$$\begin{aligned} \varepsilon_1(1, 1, 1, 1) &= (bc, -i, ib, c) \in \mathfrak{P}_1, \\ \varepsilon_2(1, 1, 1, 1) &= (ac, a, -i, ic) \in \mathfrak{P}_2, \\ \varepsilon_3(1, 1, 1, 1) &= (ab, ia, b, -i) \in \mathfrak{P}_3. \end{aligned}$$

Further details are unenlightening. ■

4.3. Equivariant realization of the point modules for A . In the rest of this section, the letter F denotes one of the modules $\tilde{V}(\tau + \xi)$ defined in §2.10. [Proposition 4.3](#) shows there are four Γ -equivariant structures on the S' -module $F \otimes k^2$. [Proposition 4.6](#) shows that the 16 ordinary point modules for A are isomorphic to $(F \otimes k^2)^\Gamma$ as F varies and each $F \otimes k^2$ takes on its four equivariant structures. Moreover, if $F = \tilde{V}(\tau + \xi_j)$ the four point modules $(F \otimes k^2)^\Gamma$ belong to \mathfrak{P}_j .

The four special point modules for A arise from the four special point modules for S in a similar way. If P is one of the four special point modules for S , then $P^{\oplus 2} \otimes k^2$ has a unique Γ -equivariant S' -module structure and $(P^{\oplus 2} \otimes k^2)^\Gamma$ is one of the 4 special point modules for A .

Proposition 4.3. *Up to isomorphism, there are four Γ -equivariant structures on the S' -module $F \otimes k^2$.*

Proof. First, we must show that $\gamma^*(F \otimes k^2) \cong F \otimes k^2$ as S' -modules for all $\gamma \in \Gamma$.

By [Proposition 2.21\(3\)](#), $\gamma^*F \cong F$ in $\text{Gr}(S)$. Since the identification of the isomorphism classes of S -modules with those of S' -modules via the Morita equivalence $- \otimes k^2$, intertwines the actions of γ^* on $\text{Gr}(S)$ and $\text{Gr}(S')$, F is a Γ -invariant S -module if and only if $F \otimes k^2$ is a Γ -invariant S' -module. Since $\gamma^*F \cong F$, i.e., since F is Γ -invariant, we can construct actions of Γ on $\text{Aut}_{S'}(F \otimes k^2) \cong k^\times \cong \text{Aut}_S(F)$ as in the proof of [9, Prop. 10.2]; both will be trivial.

Step 1: Existence of an equivariant structure. By Step 1 in the proof of [9, Prop. 10.2], the existence of an equivariant structure is controlled by an obstruction $c \in H^2(\Gamma, k^\times)$ where $H^2(\Gamma, k^\times)$ is computed with respect to the trivial Γ -action on k^\times from above.

This cohomology group is isomorphic to $\mathbb{Z}/2$ so it is not immediately clear that $c = 0$.

In the same way, there are obstructions $c_1, c_2 \in H^2(\Gamma, k^\times)$ to the existence of a Γ -equivariant structure on the S -module F and the $M_2(k)$ -module k^2 respectively.

Since c is the obstruction for $F \otimes k^2$, we get $c = c_1 + c_2$ by construction (e.g. [9, Equation (25)]). It is easy to see that c_2 is non-zero, so it remains to show that c_1 is also non-zero. In other words, the goal is to prove that the only Γ -equivariant structure on the S -module F is the trivial one. This is what [Lemma 4.4](#) below does.

Step 2: Classification of equivariant structures. As in [9, Remark 10.3], once we know an equivariant structure exists we also know that the set of isomorphism classes of such structures is acted upon simply transitively by $H^1(\Gamma, k^\times) \cong \Gamma$. ■

Lemma 4.4. *There are no Γ -equivariant structures on the S -module F .*

Proof. Suppose there were one. Then the space of equivariant structures would be a homogeneous space over $H^1 = H^1(\Gamma, k^\times) \cong \Gamma$.

If $n \geq 0$, then $\dim_k(F_n) = 2$ so F_n is a direct sum of two Γ -eigenspaces. We denote the constituent characters of the Γ -action on F_n by χ_n and η_n .

Since F is 1-critical, [22, Lemma 2.10] tells us that a central element of S either annihilates F or acts faithfully on it. By the remark after [Proposition 2.16](#), some $\Omega(z)$ acts faithfully on F . Alternatively, if all $\Omega(z)$ annihilated F , then F would be a module over $B = S/(\Omega_0, \Omega_1)$; however, by the main result in [5], the 1-critical B -modules are shifts of point modules. Since all $\Omega(z)$ are fixed by Γ and have degree two, the multi-sets $\{\chi_n, \eta_n\}$ and $\{\chi_{n+2}, \eta_{n+2}\}$ are equal.

On the one hand, equivariant structures can be twisted by characters of Γ (this is the free transitive action of H^1 on the set of equivariant structures). On the other hand though, $F(n)_{\geq 0} \cong F$ for all $n \geq 0$ so shifting the equivariant structure must produce the old equivariant structure twisted by some character. In other words, there is a character χ such that for all n we have

$$\{\chi_{n+1}, \eta_{n+1}\} = \{\chi_n + \chi, \eta_n + \chi\}.$$

The upshot of this is that there are two characters η, η' such that

$$\{\chi_{n+1}, \eta_{n+1}\} \cap \{\chi_n + \eta, \eta_n + \eta\} = \emptyset$$

for all n and similarly for η' (take η, η' to be distinct from either χ or $\eta_n - \chi_n + \chi$). In this case the two generators among x_i , $0 \leq i \leq 3$, that are η - and respectively η' -eigenvectors for the action of Γ must annihilate F . But this does not happen because all x_i act faithfully on all $V(\tau + \xi)$. ■

For each $\xi_j \in E[2]$, the four equivariant structures on $\tilde{V}(\tau + \xi_j) \otimes k^2$, give rise, by descent, to the four point modules for A that belong to \mathfrak{P}_j .

Corollary 4.5. *Fix $j \in \{0, 1, 2, 3\}$ and let $F = \tilde{V}(\tau + \xi_j)$. If $p \in \mathfrak{P}_j$, there is a unique equivariant structure on the S' -module $F \otimes k^2$ such that $M_p \cong (F \otimes k^2)^\Gamma$.*

Proof. By Proposition 4.3, there are four pairwise non-isomorphic Γ -equivariant S' -module structures on $F \otimes k^2$. Hence, by [9, Proposition 3.2], the four A -modules $(F \otimes k^2)^\Gamma$ are pairwise non-isomorphic. Since $\Omega(\tau + \xi_j)$ annihilates F and $F \otimes k^2$ it also annihilates each $(F \otimes k^2)^\Gamma$. ■

Proposition 4.6. *If P is a special point module for S , then there is a unique Γ -equivariant structure on the S' -module $(P \oplus P) \otimes k^2$.*

Proof. Recall that $P \cong S/I$ where I is an ideal generated by three of the x_i 's.

Fix elements γ and δ such that $\Gamma = \langle \gamma \rangle \times \langle \delta \rangle$. Fix an ordered basis $\{v, v'\}$ for k^2 such that the actions of γ and δ on $M_2(k)$ are conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively.

Existence. The simple structure of P will make it easy to check directly that the following construction really is an equivariant structure.

Make γ act on $(P_0 \oplus P_0) \otimes k^2 = (P_0 \oplus P_0) \otimes v \oplus (P_0 \oplus P_0) \otimes v'$ as 1 on the two outermost summands and as -1 on the two innermost ones. This ensures that the action of γ on $(P_0 \oplus P_0) \otimes k^2$ anti-commutes with the action of $1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in S \otimes M_2(k)$, as it should.

Next, let δ interchange the two outermost P_0 's and also the two innermost ones, thus making the action of δ on $(P_0 \oplus P_0) \otimes k^2$ commute with the actions of both γ and $1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This action of Γ on the degree-zero component of $(P \oplus P) \otimes k^2$ can be extended uniquely (the uniqueness follows from [9, Lemma 10.3]) to $(P \oplus P) \otimes k^2$ so as to make it compatible with the S' -action.

Uniqueness. Since γ commutes with the idempotents $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, it must implement $\langle \gamma \rangle$ -equivariant structures on the two copies M and N of $P \oplus P$ in $(P \oplus P) \otimes k^2$. Similarly, δ must interchange these two subspaces of $(P \oplus P) \otimes k^2$.

Because $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ anti-commutes with γ , it interchanges its 1 and -1 -eigenspaces. There are now two possibilities: Either M_0 and N_0 are eigenspaces for γ with opposite eigenvalues, or both break up as direct sums of one-dimensional 1 and -1 -eigenspaces. The former case is impossible because then δ , which commutes with γ , would not leave its eigenspaces invariant. This means that we are in the latter case, and we can choose decompositions $M \cong P \oplus P$ and $N \cong P \oplus P$ that make the present equivariant structure agree with the one constructed explicitly above. ■

5. COMMUTING LINES IN $\text{Gr}(A)$ AND COMMUTING SUBSPACES OF A_1

This section concerns line modules that correspond to pairs of commuting elements in A_1 .

5.1. A 2-dimensional subspace $ky + ky' \subseteq A_1$ is called a **commuting subspace** of A_1 if $[y, y'] = 0$. Such subspaces exist. For example, $[iy_0 + bcy_1, cy_2 + iby_3] = [iy_0 - bcy_1, cy_2 - iby_3] = 0$.³

If $ky + ky'$ is a commuting subspace of A_1 we call the line $y = y' = 0$ in $\mathbb{P}(A_1^*)$ a **commuting line**.

Proposition 5.2 shows that the set of commuting lines in $\mathbb{P}(A_1^*)$, and therefore the set of commuting subspaces, is parametrized by a smooth conic in the Grassmannian $\mathbb{G}(1, 3)$.

Proposition 5.1. *If $ky + ky'$ is a commuting subspace of A_1 , then $A/Ay + Ay'$ is a line module.*

³The automorphism $\gamma_2 : A \rightarrow A$ sends the commuting pair $\{iy_0 + bcy_1, cy_2 + iby_3\}$ to $\{iy_0 - bcy_1, cy_2 - iby_3\}$.

Proof. By [22, Prop.2.8], if $s, t, u, v \in A_1$ are non-zero elements such that $st = uv$ and $\{t, v\}$ is linearly independent, then $A/At + Av$ is a line module.⁴ In particular, if $ky + ky'$ is a commuting subspace of A_1 , then $yy' = y'y$ so $A/Ay + Ay'$ is a line module. ■

We call line modules of this form **commuting line modules**.

5.2. The commuting conic. The set of lines in $\mathbb{P}(A_1^*)$ is a projective variety, the Grassmanian $\mathbb{G}(1, 3)$, which we will view as a closed subvariety of $\mathbb{P}^5 = \mathbb{P}(\wedge^2 A_1^*)$. The Plücker coordinates $z_{01}, z_{02}, z_{03}, z_{12}, z_{13}, z_{23}$, on this \mathbb{P}^5 are defined as follows.

The Plücker coordinates of the line $\sum_{j=0}^3 \lambda_j y_j = \sum_{j=0}^3 \lambda'_j y_j = 0$ in $\mathbb{P}(A_1^*)$ or, more precisely, of the point in $\mathbb{G}(1, 3) \subseteq \mathbb{P}^5$ that corresponds to it, are $(M_{01}, M_{02}, M_{03}, M_{12}, M_{13}, M_{23})$ where M_{ij} is the determinant of the 2×2 sub-matrix of

$$(5-1) \quad M := \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda'_0 & \lambda'_1 & \lambda'_2 & \lambda'_3 \end{pmatrix}$$

consisting of columns i and j where the columns are labelled 0, 1, 2, 3, starting from the left. For example, $M_{01} = \lambda_0 \lambda'_1 - \lambda'_0 \lambda_1$. It is also convenient to define $M_{ji} = -M_{ij}$ and $z_{ji} = -z_{ij}$.

The Plücker coordinates satisfy the Plücker relation $z_{01}z_{23} + z_{02}z_{31} + z_{03}z_{12} = 0$. Indeed, $\mathbb{G}(1, 3)$ is the quadric hypersurface in \mathbb{P}^5 cut out by this equation.

In anticipation of the next result we call the subvariety of \mathbb{P}^5 given by (5-2) the **commuting conic**. We denote it by C_0 . It parametrizes the isomorphism classes of commuting line modules.

Proposition 5.2. *A line $y = y' = 0$ in $\mathbb{P}(A_1^*)$ is a commuting line if and only if the corresponding point in $\mathbb{G}(1, 3)$ lies on the smooth conic*

$$(5-2) \quad z_{01}z_{23} + z_{02}z_{31} + z_{03}z_{12} = z_{23} + \alpha z_{01} = z_{31} + \beta z_{02} = z_{12} + \gamma z_{03} = 0.$$

There is an isomorphism $\psi : \mathbb{P}^1 \rightarrow C_0$ given by the formulas

$$\psi(s, t) = (ia^{-1}(s^2 + t^2), 2b^{-1}st, c^{-1}(s^2 - t^2), -c(s^2 - t^2), 2bst, -ia(s^2 + t^2))$$

and $\psi^{-1}(z_{01}, \dots, z_{23}) = (cz_{03} - ia z_{01}, bz_{02})$.

Proof. Let $y = \lambda_0 y_0 + \dots + \lambda_3 y_3$, $y' = \lambda'_0 y_0 + \dots + \lambda'_3 y_3$, and let M be the matrix in (5-1). A calculation shows that

$$[y, y'] = \sum_{i,j=0}^3 \lambda_i \lambda'_j [y_i y_j] = \sum_{0 \leq i < j \leq 3} (\lambda_i \lambda'_j - \lambda'_i \lambda_j) [y_i, y_j] = \sum_{0 \leq i < j \leq 3} M_{ij} [y_i, y_j].$$

Since $[y_0, y_j] = \alpha_j [y_k, y_i]$ whenever (i, j, k) is a cyclic permutation of $(1, 2, 3)$ this sum equals

$$(\alpha M_{01} + M_{23})[y_2, y_3] + (\beta M_{02} + M_{31})[y_3, y_1] + (\gamma M_{03} + M_{12})[y_1, y_2].$$

Thus, $[y, y'] = 0$ if and only if

$$(5-3) \quad \alpha M_{01} + M_{23} = \beta M_{02} + M_{31} = \gamma M_{03} + M_{12} = 0.$$

The minors satisfy the Plücker relation so $[y, y'] = 0$ if and only if the corresponding point in $\mathbb{G}(1, 3)$ lies on the subvariety of $\mathbb{P}^5 = \mathbb{P}(\wedge^2 A_1^*)$ given by (5-2). We complete the proof by showing this subvariety is a smooth conic.

The equations $z_{23} + \alpha z_{01} = z_{31} + \beta z_{02} = z_{12} + \gamma z_{03} = 0$ cut out a \mathbb{P}^2 in \mathbb{P}^5 so the commuting conic is the zero locus of the quadratic form $z_{01}z_{23} + z_{02}z_{31} + z_{03}z_{12}$ on this \mathbb{P}^2 . By using the coordinate functions z_{01}, z_{02}, z_{03} , for this \mathbb{P}^2 and making the substitutions $z_{23} = -\alpha z_{01}$, $z_{13} = \beta z_{02}$, and $z_{12} = -\gamma z_{03}$, this quadratic form becomes $\alpha z_{01}^2 + \beta z_{02}^2 + \gamma z_{03}^2$ which is non-degenerate because $\alpha\beta\gamma \neq 0$. Thus, the commuting conic is smooth as claimed.

⁴The result in [22, Prop.2.8] is stated for a class of algebras that does not include A but its proof depends only on the good homological properties of the ring, its Hilbert series, and the fact that the ring in question is generated in degree one, so the result applies to A .

We leave the reader to check that ψ is an isomorphism with the claimed inverse. \blacksquare

Proposition 5.3. *The commuting conic is cut out by the equations $z_{23} + \alpha z_{01} = z_{31} + \beta z_{02} = z_{12} + \gamma z_{03} = 0$ and any one of the following four equations:*

$$\begin{aligned} \alpha z_{01}^2 + \beta z_{02}^2 + \gamma z_{03}^2 &= 0, & \text{or} \\ \alpha\beta\gamma z_{01}^2 + \beta z_{12}^2 + \gamma z_{13}^2 &= 0, & \text{or} \\ \alpha\beta\gamma z_{02}^2 + \alpha z_{12}^2 + \gamma z_{23}^2 &= 0, & \text{or} \\ \alpha\beta\gamma z_{01}^2 + \alpha z_{13}^2 + \beta z_{23}^2 &= 0. \end{aligned}$$

Proof. One obtains these equations by using the equations $z_{23} + \alpha z_{01} = z_{31} + \beta z_{02} = z_{12} + \gamma z_{03} = 0$ and making the obvious substitutions into the equation $z_{01}z_{23} + z_{02}z_{31} + z_{03}z_{12} = 0$. \blacksquare

Proposition 5.4. *The commuting lines $y = y' = 0$ in $\mathbb{P}(A_1^*)$ are precisely the lines in one of the rulings on the quadric $y_0^2 + \beta\gamma y_1^2 + \gamma\alpha y_2^2 + \alpha\beta y_3^2 = 0$.*

Proof. The equation for the quadric is $(y_0 + ibcy_1)(y_0 - ibcy_1) = \alpha(icy_2 + by_3)(icy_2 - by_3)$. For all $t \in \mathbb{P}^1$, the line

$$(5-4) \quad (y_0 + ibcy_1) - t\alpha(icy_2 - by_3) = t(y_0 - ibcy_1) - \alpha(icy_2 + by_3) = 0$$

lies on the quadric. As t varies over \mathbb{P}^1 one obtains all the lines in one of the rulings. The Plücker coordinates of the line (5-4) are given by the 2×2 minors of the matrix

$$\begin{pmatrix} 1 & ibc & -iact & abt \\ t & -ibct & -iac & -ab \end{pmatrix}.$$

The corresponding point in $\mathbb{G}(1, 3)$, namely

$$(-2ibct, iac(t^2 - 1), -ab(t^2 + 1), ab\gamma(t^2 + 1), iac\beta(t^2 - 1), 2iabct),$$

lies on the plane $z_{23} + \alpha z_{01} = z_{31} + \beta z_{02} = z_{12} + \gamma z_{03} = 0$ and hence on the commuting conic. \blacksquare

Proposition 5.5. *Let $p \in \mathfrak{P}$.*

- (1) *The point p lies on the quadric in Proposition 5.4 if and only if $p \in \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3$.*
- (2) *If $p \in \mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3$, there is a unique commuting line passing through p .*
- (3) *None of the points in $\mathfrak{P}_\infty \cup \mathfrak{P}_0$ lies on a commuting line.*

Proof. Part (1) is a simple calculation. Parts (2) and (3) follow from the fact that a smooth quadric in \mathbb{P}^3 is the disjoint union of the lines in one of the rulings on it. \blacksquare

5.3. Commuting subspaces of A_1 . Just as the commuting lines provide a ruling on the quadric $y_0^2 + \beta\gamma y_1^2 + \gamma\alpha y_2^2 + \alpha\beta y_3^2 = 0$ in $\mathbb{P}(A_1^*)$, the commuting subspaces of A_1 provide a ruling on a quadric in $\mathbb{P}(A_1)$. We determine that quadric after the following routine lemma.

Let V be a finite dimensional vector space and V^* its dual. If L is a linear subspace of $\mathbb{P}(V)$ we write L^\perp for the linear subspace of $\mathbb{P}(V^*)$ that vanishes on L .

Lemma 5.6. *Let $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in k^\times$. Let V be a 4-dimensional vector space with basis y_0, \dots, y_3 . Let w_0, \dots, w_3 be the dual basis for V^* . Let $S \subseteq \mathbb{P}(V^*)$ and $S' \subseteq \mathbb{P}(V)$ be the smooth quadrics defined by $\lambda_0^2 y_0^2 + \lambda_1^2 y_1^2 + \lambda_2^2 y_2^2 + \lambda_3^2 y_3^2 = 0$ and $\lambda_0^{-2} w_0^2 + \lambda_1^{-2} w_1^2 + \lambda_2^{-2} w_2^2 + \lambda_3^{-2} w_3^2 = 0$.*

- (1) *For each $t \in \mathbb{P}^1$, the lines*

$$L_t : (\lambda_0 y_0 + i\lambda_1 y_1) - t(i\lambda_2 y_2 + \lambda_3 y_3) = t(\lambda_0 y_0 - i\lambda_1 y_1) - (i\lambda_2 y_2 - \lambda_3 y_3) = 0 \quad \text{and}$$

$$L'_t : (\lambda_0^{-1} w_0 + i\lambda_1^{-1} w_1) - t(i\lambda_2^{-1} w_2 + \lambda_3^{-1} w_3) = t(\lambda_0^{-1} w_0 - i\lambda_1^{-1} w_1) - (i\lambda_2^{-1} w_2 - \lambda_3^{-1} w_3) = 0$$

lie on S and S' respectively.

- (2) *$L'_t = L_t^\perp$.*

Proof. It becomes obvious that L_t lies on S when we write the equation for S as

$$(\lambda_0 y_0 + i\lambda_1 y_1)(\lambda_0 y_0 - i\lambda_1 y_1) = (i\lambda_2 y_2 + \lambda_3 y_3)(i\lambda_2 y_2 - \lambda_3 y_3).$$

The points $(\lambda_0^{-1}, i\lambda_1^{-1}, -it\lambda_2^{-1}, -t\lambda_3^{-1})$ and $(-t\lambda_0^{-1}, it\lambda_1^{-1}, i\lambda_2^{-1}, -\lambda_3^{-1})$ lie on L_t .

Similarly, the line L'_t lies on S' and passes through the two points $(\lambda_0, i\lambda_1, -it\lambda_2, -t\lambda_3)$ and $(-t\lambda_0, it\lambda_1, i\lambda_2, -\lambda_3)$, both of which belong to

$$(\lambda_0^{-1}, i\lambda_1^{-1}, -it\lambda_2^{-1}, -t\lambda_3^{-1})^\perp \cap (-t\lambda_0^{-1}, it\lambda_1^{-1}, i\lambda_2^{-1}, -\lambda_3^{-1})^\perp = L_t^\perp.$$

Thus $L'_t \subseteq L_t^\perp$ and, for dimension reasons, $L'_t = L_t^\perp$. \blacksquare

Proposition 5.7. *Let w_0, w_1, w_2, w_3 be the basis for A_1^* dual to the basis y_0, y_1, y_2, y_3 . The commuting subspaces of A_1 form a ruling on the quadric $\alpha\beta\gamma w_0^2 + \alpha w_1^2 + \beta w_2^2 + \gamma w_3^2 = 0$ in $\mathbb{P}(A_1)$. The commuting subspace vanishing on the commuting line in (5-4) is*

$$(5-5) \quad (abcw_0 + iaw_1) - t(ibw_2 - cw_3) = t(abcw_0 - iaw_1) - (ibw_2 + cw_3) = 0.$$

Proof. In order that the line in (5-4) be the line L_t in Lemma 5.6, we must take $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (1, bc, ac, -ab)$. Hence $(\lambda_0^{-1}, \lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}) = (abc)^{-1}(abc, a, b, -c)$. The line L_t^\perp , i.e., the commuting subspace vanishing on L_t , is the line L'_t in Lemma 5.6 so is given by (5-5). \blacksquare

Lemma 5.8. *There are no commuting subspaces in S_1 .*

Proof. Let $x = \varepsilon x_0 + \lambda x_1 + \mu x_2 + \nu x_3$ and $x' = \varepsilon' x_0 + \lambda' x_1 + \mu' x_2 + \nu' x_3$. We use the same notation as in Proposition 5.2 for the 2×2 minors of

$$M = \begin{pmatrix} \varepsilon & \lambda & \mu & \nu \\ \varepsilon' & \lambda' & \mu' & \nu' \end{pmatrix}.$$

A computation gives

$$\begin{aligned} [x, x'] &= M_{01}[x_0, x_1] + M_{02}[x_0, x_2] + M_{03}[x_0, x_3] + M_{12}\{x_0, x_3\} - M_{13}\{x_0, x_2\} + M_{23}\{x_0, x_1\} \\ &= (M_{01} + M_{23})x_0x_1 + (M_{23} - M_{01})x_1x_0 + (M_{02} - M_{13})x_0x_2 - (M_{13} + M_{02})x_2x_0 \\ &\quad + (M_{03} + M_{12})x_0x_3 + (M_{12} - M_{03})x_3x_0 \end{aligned}$$

The relations for S can be written as

$$x_j x_k = \frac{1}{2}(\alpha_i^{-1}[x_0, x_i] + \{x_0, x_i\}) \quad x_k x_j = \frac{1}{2}(\alpha_i^{-1}[x_0, x_i] - \{x_0, x_i\})$$

as (i, j, k) runs over all cyclic permutations of $(1, 2, 3)$ so $\{x_j^2, x_0 x_i, x_i x_0 \mid 1 \leq i \leq 3, 0 \leq j \leq 3\}$ is a basis for S_2 . Hence if $[x, x'] = 0$, then all the 2×2 minors of M vanish. This implies that x and x' are linearly dependent. Therefore S_1 does not contain a commuting subspace. \blacksquare

In retrospect it is clear why A_1 but not S_1 contains commuting subspaces. The relations for A are such that the subspace of A_2 spanned by the commutators $\{[y, y'] \mid y, y' \in A_1\}$ has dimension 3 whereas the subspace of S_2 spanned by $\{[x, x'] \mid x, x' \in S_1\}$ has dimension 6. This is due to the fact that A has relations $[y_0, y_i] = \alpha_i[y_j, y_k]$ but S has relations $[x_0, x_i] = \alpha_i\{x_j, x_k\}$.

6. MORE CONIC LINE MODULES VIA QUANTUM SYMMETRIES

6.1. The commuting conic, C_0 , parametrizing the commuting line modules is “biased” as far as point-line incidence is concerned: by Proposition 5.5, only points in the families $\mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \mathfrak{P}_3$ lie on a commuting line. This section cures this bias by finding three more conics $C_1, C_2, C_3 \subseteq \mathbb{G}(1, 3)$ that parametrize line modules with the property that every point in $\mathfrak{P} - (\mathfrak{P}_\infty \cup \mathfrak{P}_j)$ lies on exactly one line in C_j and no points in $\mathfrak{P}_\infty \cup \mathfrak{P}_j$ lie on a line in C_j .

The new conics will be obtained by “moving C_0 around”. One possible meaning for this phrase would be to take an automorphism $\psi \in \text{Aut}(A)$ and send a line module L to ψ^*L . This does not do what we want because ψ sends commuting subspaces to commuting subspaces whence ψ^*L belongs

to C_0 if L does. In other words, A does not have enough symmetries to “move” C_0 . Nevertheless, we will show that A has enough “quantum symmetries” for this purpose, in a sense that will be made precise below. The quantum symmetries are, in effect, auto-equivalences of $\text{Gr}(A)$.

6.2. When G is a finite group we write $k(G)$ for the algebra of k -valued functions on G and kG for its group algebra; $k(G)$ and kG are mutually dual Hopf algebras.

The construction of A from S can be described in the following way. First regard S as a $k(H_4)$ -comodule algebra via the homomorphism $H_4 \rightarrow \text{Aut}(S)$ defined in Proposition 2.7. In other words, S is an algebra object in the category $\mathcal{M}^{k(H_4)}$ of right $k(H_4)$ -comodules. Then apply to S the monoidal functor $\mathcal{M}^{k(H_4)} \rightarrow \text{VECT}$ corresponding to the 2-cocycle $\sigma : k(H_4)^{\otimes 2} \rightarrow k$ that is the composition of the homomorphism $k(H_4)^{\otimes 2} \rightarrow k(\Gamma)^{\otimes 2}$ induced by the inclusion $\Gamma = \langle \varepsilon_1^2, \varepsilon_2^2 \rangle \subseteq H_4$ with the cocycle $\mu : k(\Gamma)^{\otimes 2} = (k\hat{\Gamma})^{\otimes 2} \rightarrow k$ in §3.1.2. By monoidality, the image of S under this functor is an algebra in VECT , namely A .

By the general formalism of how cocycles classify such monoidal functors (as covered in [6], say), our functor $\mathcal{M}^{k(H_4)} \rightarrow \text{VECT}$ factors as

$$(6-1) \quad \begin{array}{ccc} \mathcal{M}^{k(H_4)} & \xrightarrow{\quad} & \text{VECT} \\ \searrow \equiv & & \nearrow \text{forget} \\ & \mathcal{M}^H & \end{array}$$

where H is defined below and all arrows are monoidal functors and $\mathcal{M}^{k(H_4)} \rightarrow \mathcal{M}^H$ is a monoidal equivalence, as indicated.

6.3. **The Hopf algebra H .** The “quantum symmetries” of A alluded to in the title of this section will be implemented by a new Hopf algebra H which is a “deformation” of the Hopf algebra $k(H_4)$ by the cocycle σ . The construction is due to Doi [17]:

- (1) as a vector space, $H = k(H_4)$;
- (2) the coalgebra structure on H is exactly the same as that on $k(H_4)$;
- (3) as an algebra, H is $k(H_4)$ with the new multiplication

$$r * s := \sigma^{-1}(r_1 \otimes s_1) \sigma(r_3 \otimes s_3) r_2 s_2,$$

where rs is the product in $k(H_4)$, $\sigma^{-1} : k(H_4)^{\otimes 2} \rightarrow k$ is the convolution-inverse of σ , and we are using Sweedler’s convention $(\Delta \otimes \text{id})\Delta(r) = r_1 \otimes r_2 \otimes r_3$ with an implied summation.

As we will see more explicitly below, H is not commutative so is not isomorphic as a Hopf algebra to $k(G)$ for any group G . Instead, we think of H as *the ring of k -valued functions on a finite quantum group* whose action as “automorphisms” of A is made manifest by an algebra homomorphism $A \rightarrow A \otimes H$ that implements the quantum symmetries in the title of this section.

Since A is obtained from S by deforming via the cocycle σ , A is the image of S through the horizontal map in (6-1), and hence an H -comodule algebra (see [6, pp. 25-26] for more about this).

6.4. We now explain how these quantum symmetries (i.e., the action of the quantum group on $\text{Gr}(A)$) produce(s) new line modules.

Let M be a left A -module and V a left H -module. Then $M \otimes V$ is a left $A \otimes H$ -module and therefore a left A -module via the comodule structure map $A \rightarrow A \otimes H$. If M is graded so is $M \otimes V$ with $\deg(M_n \otimes V) = n$. Specializing to the graded case, we get an action

$$(6-2) \quad \text{Gr}(A) \boxtimes \text{Mod}(H) \rightarrow \text{Gr}(A)$$

of the monoidal category $\text{Mod}(H)$ on the category $\text{Gr}(A)$ for a categorical tensor product construction \boxtimes that we will not make precise here.

6.4.1. Let $\gamma : H \rightarrow k$ be an algebra homomorphism and let k_γ be the corresponding 1-dimensional H -module. Since γ is a group-like element in the dual Hopf algebra H^* it implements an automorphism, φ_γ say, of A as a graded algebra. The action of k_γ on $\text{Gr}(A)$ via (6-2) is the same as the auto-equivalence that twists every module and morphism by φ_γ . We have already observed that such a twist applied to a line module in C_0 produces another line module in C_0 , so that there is no chance of discovering new families of line modules in this fashion. However, since H is not commutative, we might hope that for a line module $M \in \text{Gr}(A)$ corresponding to a point in the commuting conic C_0 and some simple H -module V of dimension ≥ 2 the tensor product $M \otimes V$ (obtained by acting with $V \in \text{Mod}(H)$ on $M \in \text{Gr}(A)$ via (6-2)) might break up (decompose) as a direct sum of several line modules in these other as-yet hypothetical families C_j , $1 \leq j \leq 3$. We will see presently that this is indeed what happens.

6.5. We first need a better understanding of the algebra structure on H . It is simpler to do this dually by understanding the co-multiplication on the dual Hopf algebra H^* instead; we will then freely switch points of view between the equivalent categories of left H -modules and right H^* -comodules. In (6-2), for instance, we might substitute \mathcal{M}^{H^*} for $\text{Mod}(H)$.

Since $H = k(H_4)$ we can, and will, use the usual isomorphism $kH_4 \rightarrow k(H_4)^*$ to identify H^* , as a vector space, with the group algebra kH_4 . We will therefore write $H^* = kH_4$. Although the coproducts on H^* and kH_4 are different the relation between them is quite simple.

Lemma 6.1. *Let $\xi_i = \varepsilon_i^2$. The coproduct on H^* is the usual group algebra coproduct on kH_4 followed by conjugation in $kH_4 \otimes kH_4$ by the involutive element*

$$J := \frac{1}{2}(1 \otimes 1 + \xi_1 \otimes 1 + 1 \otimes \xi_2 - \xi_1 \otimes \xi_2) \in k\Gamma^{\otimes 2} \subseteq kH_4^{\otimes 2}.$$

Proof. This can be seen by using the explicit form of the cocycle on $k(\Gamma) = k\hat{\Gamma}$ in §3.1.2. ■

Using the fact that conjugation by ξ_2 multiplies ε_1 by $\eta = \delta^2$ and $\xi_1 = \varepsilon_1^2$ commutes with ε_1 , the comultiplication $\Delta : H^* \rightarrow H^* \otimes H^*$ is such that

$$\begin{aligned} \Delta(\varepsilon_1) &= \frac{1}{2}(\varepsilon_1 \otimes \varepsilon_1 + \varepsilon_1 \otimes \varepsilon_1 \eta + \varepsilon_1^{-1} \otimes \varepsilon_1 - \varepsilon_1^{-1} \otimes \varepsilon_1 \eta) \\ \Delta(\varepsilon_1 \eta) &= \frac{1}{2}(\varepsilon_1 \eta \otimes \varepsilon_1 \eta + \varepsilon_1 \eta \otimes \varepsilon_1 + \varepsilon_1^{-1} \eta \otimes \varepsilon_1 \eta - \varepsilon_1^{-1} \eta \otimes \varepsilon_1) \\ \Delta(\varepsilon_1^{-1}) &= \frac{1}{2}(\varepsilon_1^{-1} \otimes \varepsilon_1^{-1} + \varepsilon_1^{-1} \otimes \varepsilon_1^{-1} \eta + \varepsilon_1 \otimes \varepsilon_1^{-1} - \varepsilon_1 \otimes \varepsilon_1^{-1} \eta) \\ \Delta(\varepsilon_1^{-1} \eta) &= \frac{1}{2}(\varepsilon_1^{-1} \eta \otimes \varepsilon_1^{-1} \eta + \varepsilon_1^{-1} \eta \otimes \varepsilon_1^{-1} + \varepsilon_1 \eta \otimes \varepsilon_1^{-1} \eta - \varepsilon_1 \eta \otimes \varepsilon_1^{-1}) \end{aligned}$$

Lemma 6.2. *The subspaces*

$$\begin{aligned} D_1 &:= \text{span}\{\varepsilon_1^{\pm 1}, \varepsilon_1^{\pm 1} \eta\}, \\ D_2 &:= \text{span}\{\varepsilon_2^{\pm 1}, \varepsilon_2^{\pm 1} \eta\}, \\ D_3 &:= \text{span}\{\varepsilon_1 \varepsilon_2, \varepsilon_1^{-1} \varepsilon_2^{-1}, \varepsilon_1 \varepsilon_2 \eta, \varepsilon_1^{-1} \varepsilon_2^{-1} \eta\}, \end{aligned}$$

of H^ are sub-coalgebras and each is isomorphic to $M_2(k)$ as a coalgebra. For example, the elements*

$$(6-3) \quad \begin{cases} a := \frac{1}{2}(\varepsilon_1 + \varepsilon_1 \eta) & b := \frac{1}{2}(\varepsilon_1 - \varepsilon_1 \eta) \\ c := \frac{1}{2}(\varepsilon_1^{-1} - \varepsilon_1^{-1} \eta) & d := \frac{1}{2}(\varepsilon_1^{-1} + \varepsilon_1^{-1} \eta) \end{cases}$$

are matrix co-units for D_1 in the sense that the comultiplication Δ on $M_2(H^)$ has the property that*

$$(6-4) \quad \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Lemma 6.3. *The restriction of the comultiplication on H^* to $k\langle \varepsilon_1^2, \varepsilon_2^2, \delta \rangle \subseteq kH_4 = H^*$ is the usual group algebra comultiplication.*

Proof. The subgroup $G := \langle \varepsilon_1^2 = \xi_1, \varepsilon_2^2 = \xi_2, \delta \rangle$ of H_4 is abelian. Since J belongs to $kG \otimes kG$ it commutes with ε_1^2 , ε_2^2 , and δ . Now apply [Lemma 6.2](#). \blacksquare

Multiplying D_1 , D_2 and D_3 in H^* by elements of $\langle \varepsilon_1^2, \varepsilon_2^2, \delta \rangle$ we get translates that are again 2×2 matrix subcoalgebras of H . It is easy to see that each D has four such translates, and the twelve translates are pairwise distinct.

Proposition 6.4. *As a coalgebra, H^* is a direct sum of 16 one-dimensional coalgebras spanned by the elements in $\langle \varepsilon_1^2, \varepsilon_2^2, \delta \rangle$ and twelve 2×2 matrix subcoalgebras. In particular, H^* is cosemisimple.*

We could have seen that H^* is cosemisimple from general principles which ensure that cosemisimplicity is preserved under the kind of twisting procedure by which H^* was obtained from kH_4 .

6.6. The comodules V_j and the endofunctors $\bullet \otimes V_j$ of $\text{Gr}(A)$. For $j = 1, 2, 3$, there is a unique two-dimensional right D_j -comodule up to isomorphism. We will describe one such D_1 -comodule (equivalently, D_1^* -module) which we will denote by V_1 .

Let $\{a^*, b^*, c^*, d^*\}$ be the basis for $D_1^* \subset H$ that is dual to the basis $\{a, b, c, d\}$ for D_1 in (6-3). They are matrix units for the algebra D_1^* in the sense that the linear map

$$\begin{aligned} a^* &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & b^* &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ c^* &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & d^* &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

is an algebra isomorphism $D_1^* \rightarrow M_2(k)$. We define V_1 to be the left ideal of D_1^* ,

$$V_1 := ka^* + kc^*.$$

We define V_2 and V_3 in a similar way.

The functor $\bullet \otimes V_j : \text{Gr}(A) \rightarrow \text{Gr}(A)$ is the functor $M \rightsquigarrow M \otimes V_j$ where $M \otimes V_j$ is made into an A -module through the comodule structure map $A \rightarrow A \otimes H$.

Our goal, as hinted at before, is to show that if M is a commuting line module, then $M \otimes V_j$ is a direct sum of two line modules each of which belongs to a family of line modules parametrized by some other conic C_j . The next result is the first step in this direction.

Proposition 6.5. *Let $M \in \text{Gr}(A)$ and $j \in \{1, 2, 3\}$. If $\Omega(z)$ annihilates M , then $\Omega(z + \xi_j)$ annihilates $M \otimes V_j$.*

Proof. By [Corollary 2.10](#), the automorphism $\varepsilon_1 \in H_4$ sends $\Omega(z + \xi_1)$ to $\Omega(z)$, as do $\varepsilon_1 \eta$, ε_1^{-1} and $\varepsilon_1^{-1} \eta$. Thinking of A and H^* as having the same underlying vector spaces as S and kH_4 respectively, the previous sentence applies to them as well. In other words, the coaction $\rho : A \rightarrow A \otimes H$ sends $\Omega(z + \xi_1) \in A$ to

$$(6-5) \quad \Omega(z) \otimes (\text{some element in the matrix algebra summand } D_1^* \text{ of } H) + \dots$$

where the dots stand for terms whose right hand tensorands lie in other matrix algebra summands of H . Since H acts on V_1 via the surjection $H \rightarrow D_1^*$, the terms after the $+$ sign in (6-5) act as zero on $M \otimes V_1$. But the left-hand tensorand $\Omega(z)$ annihilates M by assumption, so $\Omega(z + \xi_1)$ annihilates $M \otimes V_1$.

The same argument works for $M \otimes V_2$ and $M \otimes V_3$. \blacksquare

The following decomposition result describes $M \otimes V_j$ in more detail. Before stating it, recall that $k\langle \varepsilon_1^2, \varepsilon_2^2, \delta \rangle \subset H^*$ is a subcoalgebra with its usual group algebra comultiplication, so the subgroup $\langle \varepsilon_1^2 = \xi_1, \varepsilon_2^2 = \xi_2, \delta \rangle < H_4$ acts on A by graded algebra automorphisms. In particular, $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ does too so we can define the twist autoequivalences γ_i^* on $\text{Mod}(A)$ or $\text{Gr}(A)$ as in [9, §7.2]. We will use the identification $\Gamma = E[2]$ with $\gamma_j = \xi_j$.

Proposition 6.6. *If $M \in \text{Gr}(A)$, then $M \otimes V_1 \cong N \oplus \xi_1^* N$ for some $N \in \text{Gr}(A)$.*

Proof. We will write $\mathbb{Z}/2$ for the subgroup $\langle \eta = \delta^2 \rangle \subseteq H_4$. Since δ acts as multiplication by i on A_1 , η acts on A as multiplication by $(-1)^n$ on A_n . A coaction by the function algebra $k(\mathbb{Z}/2)$ is the same thing as a $\mathbb{Z}/2$ -action, so the map

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{\text{projection} \otimes \text{id}} k(\mathbb{Z}/2) \otimes H$$

gives an action of $\mathbb{Z}/2$ as algebra automorphisms of H . Thus, $A \otimes H$ is $\mathbb{Z}/2$ -module algebra.

We now make $M \otimes V_1$ a $\mathbb{Z}/2$ -equivariant $A \otimes H$ -module: we make M a $\mathbb{Z}/2$ -equivariant A -module by having $\eta \in \mathbb{Z}/2$ act on M_n as multiplication by $(-1)^n$, and make V_1 a $\mathbb{Z}/2$ -equivariant H -module by having η fix a^* and send c^* to $-c^*$.

The image of the map $A \rightarrow A \otimes H$ is contained in the cotensor product

$$A \square_H H \subseteq A \square_{k(\mathbb{Z}/2)} H = (A \otimes H)^\eta$$

so the action of η on $M \otimes V_1$ commutes with the A -action on it. Hence the $(+1)$ - and (-1) -eigenspaces of η are graded A -submodules of $M \otimes V_1$. Finally, the involution $b^* + c^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(k) \cong D_1^* \subset H$ interchanges these eigenspaces and commutes with the action of ξ_1 on A ; i.e.,

$$(b^* + c^*)x = \xi_1(x)(b^* + c^*) \text{ on } M \otimes V_1 \text{ for all } x \in A.$$

This shows that the two η -eigenspaces are, as A -modules, twists of each other by the automorphism ξ_1 . We denote one of these eigenspaces by N ; the other is then isomorphic to $\xi_1^* N$. ■

There are analogues of [Proposition 6.6](#) with V_2 and ξ_2 (or V_3 and $\xi_1 \xi_2$) in place of V_1 and ξ_1 .

6.7. The autoequivalences a_ε of $\text{Gr}(A)$. Let $\varepsilon_1 \in H_4$ be the automorphism of S defined in [Proposition 2.7](#). We define the auto-equivalence a_{ε_1} of $\text{Gr}(A)$ by declaring that

$$(6-6) \quad a_{\varepsilon_1}(M) := (M \otimes V_1)^\eta,$$

the η -invariant subspace, or $(+1)$ -eigenspace, for the action of η defined in the proof of [Proposition 6.6](#); i.e., η acts diagonally with η acting on M_n as multiplication by $(-1)^n$ and on V_1 by fixing a^* and sending c^* to $-c^*$. Thus, $a_{\varepsilon_1}(M)$ is one of the summands in [Proposition 6.6](#).

Since η fixes a^* and changes the sign of c^* , the underlying graded vector space of $a_{\varepsilon_1}(M)$ is

$$(6-7) \quad \bigoplus_{\text{even } n} M_n \otimes \langle a^* \rangle \oplus \bigoplus_{\text{odd } n} M_n \otimes \langle c^* \rangle.$$

The algebra A acts on this graded vector space through scalar restriction via the comodule structure map $A \rightarrow A \otimes H$. As a coalgebra, H is isomorphic to the function algebra $k(H_4)$. The restriction $A_1 \rightarrow A_1 \otimes H$ is the comodule structure map corresponding to the action of H_4 as linear automorphisms of A_1 . Thus, if $y \in A_1$, then its image through this structure map is

$$\varepsilon_1(y) \otimes b^* + \varepsilon_1^{-1}(y) \otimes c^* + \dots$$

where $\varepsilon_1(y)$ is the result of the action of H_4 on A_1 dual to the $H \cong k(H_4)$ -coaction (equivalently, it is the action of H_4 on S_1 transported to A_1 through our identification $A_1 \cong S_1$) and the dots represent summands that vanish on $a_{\varepsilon_1}(M)$ because their right hand tensorands vanish on V_1 .

To see that a_{ε_1} is an equivalence we note that a quasi-inverse to it is the functor defined by the formula (6-6) with V_1 having its “other” $\mathbb{Z}/2$ -equivariant structure, i.e., the action of η that fixes c^* and sends a^* to $-a^*$.

By [Proposition 6.6](#), a_{ε_1} preserves dimensions of homogeneous components.

In §2.3 we saw that each ordinary point module associated to \mathfrak{P}_j is annihilated by $\Omega(\tau + \xi_j)$. This allows us to define the following relations on the four-element set of ordinary families.

Definition 6.7. Let $\xi \in E[2]$. Two ordinary families \mathfrak{P}_i and \mathfrak{P}_j of point modules are ξ -related if they are annihilated by $\Omega(z)$ and $\Omega(z + \xi)$ respectively for some $z \in E$. ♦

By [Proposition 4.2](#), the action of H_4 on $\mathbb{P}(A_1^*)$ is such that ε_i sends \mathfrak{P}_j to \mathfrak{P}_k where k is determined by the requirement that $2\varepsilon_1 + \xi_j = \xi_k$.

As a consequence of the proof of [Proposition 6.6](#) we get

Corollary 6.8. *For each 2-torsion point $\xi \in E$ there are four order-4 autoequivalences a_ε of $\text{Gr}(A)$ that preserve Hilbert series, permute the special point modules, and interchange the point modules in any two ξ -related ordinary families.*

Proof. Let $\xi = \xi_1$ and $\varepsilon = \varepsilon_1$.

Since a_ε sends A to a module isomorphic to A , a_ε sends cyclic modules to cyclic modules. By [Proposition 6.5](#), a_ε sends the point modules in an ordinary family to the point modules in the ξ -related family and permutes the four special point modules since each of those is annihilated by exactly one central element $\Omega(\omega)$, $\omega \in E[2]$.

The four auto-equivalences in $\{\gamma^* a_\varepsilon \mid \gamma \in \Gamma\}$ have the same permutation properties because each \mathfrak{P}_j is stable under the action of Γ .

The argument for other ξ is analogous, substituting V_2 or V_3 for V_1 . ■

Corollary 6.9. *The autoequivalence a_ε sends cyclic modules to cyclic modules and, since it preserves Hilbert series, it induces automorphisms of the point and line schemes for A .*

Proof. We noticed in the proof of [Corollary 6.8](#) that a_ε sends cyclic modules to cyclic modules. Since the point and line schemes of A classify isomorphism classes of cyclic modules with certain Hilbert series (e.g., [27]), a_ε induces an automorphism of these schemes. ■

We can say more.

Proposition 6.10. *The autoequivalences a_ε generate an action of $(\mathbb{Z}/4)^2 \cong E[4]$ on $\text{Gr}(A)$.*

Proof. By construction, the underlying graded vector space of $a_{\varepsilon_1}(M)$ is

$$(6-8) \quad \bigoplus_{\text{even } n} M_n \otimes \langle a^* \rangle \oplus \bigoplus_{\text{odd } n} M_n \otimes \langle c^* \rangle$$

and $y \in A_1$ acts on it as $\varepsilon_1(y) \otimes b^* + \varepsilon_1^{-1}(y) \otimes c^* + \dots$. Since

$$\begin{pmatrix} b^* \\ c^* \end{pmatrix} \begin{pmatrix} a^* & c^* \end{pmatrix} = \begin{pmatrix} 0 & a^* \\ c^* & 0 \end{pmatrix},$$

the action of y on the left hand tensorands of the even components in (6-8) is the initial action of $\varepsilon_1^{-1} \triangleright y$. Similarly, its action on the left hand tensorands of the odd components is the initial action of $\varepsilon_1 \triangleright y$. In conclusion, identifying (6-8) with M by

$$\langle a^* \rangle \cong k \cong \langle c^* \rangle, \quad a^* = c^* = 1,$$

$y \in A_1$ acts as $\varepsilon_1^{-1}(y)$ on the even-degree components of $a_{\varepsilon_1}(M)$ and as $\varepsilon_1(y)$ on the odd-degree components.

The same discussion applies to the other autoequivalences defined above and shows that they do indeed comprise an action of $(\mathbb{Z}/4)^2$ on $\text{Gr}(A)$, with a_{ε_1} and a_{ε_2} as generators. The fact that a_{ε_1} and a_{ε_2} commute amounts to their commutator scaling the homogeneous components of a graded module as in [Lemma 2.20\(1\)](#), and hence acting trivially. ■

Now, as before, the action of $E[4]$ on $\text{Gr}(A)$ induces one on point and line schemes. In addition, since $\mathbb{G}(1, 3)$ parametrizes the isomorphism classes of cyclic graded A -modules with Hilbert series $1 + 2t$, the $E[4]$ -action on the line scheme extends to one on $\mathbb{G}(1, 3)$.

Proposition 6.11. *The images of the conic $C_0 \subset \mathbb{P}^5$ under the action of $E[4]$ on $\mathbb{G}(1, 3)$ are conics. As subvarieties of $\mathbb{G}(1, 3)$ the four different images, C_0, C_1, C_2, C_3 , are cut out by the Plücker relation $z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12} = 0$ together with three linear relations which are:*

$$\begin{aligned} C_0 : \quad & z_{23} + \alpha z_{01} = z_{31} + \beta z_{02} = z_{12} + \gamma z_{03} = 0; \\ C_1 : \quad & z_{23} - \alpha z_{01} = z_{31} - z_{02} = z_{12} + z_{03} = 0; \\ C_2 : \quad & z_{23} + z_{01} = z_{31} - \beta z_{02} = z_{12} - z_{03} = 0; \\ C_3 : \quad & z_{23} - z_{01} = z_{31} + z_{02} = z_{12} - \gamma z_{03} = 0. \end{aligned}$$

These conics are pairwise disjoint.

Proof. By the remark before the proposition the action of $E[4]$ on $\mathbb{G}(1, 3)$ preserves the line scheme.

The Plücker embedding $\mathbb{G}(1, 3) \rightarrow \mathbb{P}^5$ is effected via the top exterior power \mathcal{L} on $\mathbb{G}(1, 3)$ of the dual tautological rank-two bundle on $\mathbb{G}(1, 3)$. Since the Picard group of $\mathbb{G}(1, 3)$ is \mathbb{Z} and $[\mathcal{L}]$ generates it [18, Theorem 2.10], every automorphism of $\mathbb{G}(1, 3)$ leaves $[\mathcal{L}]$ invariant.

In particular, $[\mathcal{L}]$ is invariant under the action of $E[4]$ on the Picard group of $\mathbb{G}(1, 3)$ so it admits an equivariant structure for the action of the universal central extension H_4 of $E[4] \cong (\mathbb{Z}/4)^2$ on $\mathbb{G}(1, 3)$. The equivariant structure allows us to lift the H_4 -action on $\mathbb{G}(1, 3)$ to an action on the ambient space \mathbb{P}^5 of the Plücker embedding. Being a conic is a statement about how C_0 sits inside \mathbb{P}^5 , so an element of H_4 sends C_0 to another conic.

The H_4 action on A_1 is obtained by transferring the H_4 action on S_1 to A_1 by using the linear isomorphism $\iota : A_1 \rightarrow S_1$

$$\iota(y_0) = x_0, \quad \iota(y_1) = -ix_1, \quad \iota(y_2) = -ix_2, \quad \iota(y_3) = x_3$$

in §3.2. Let ϕ_j , $j = 1, 2, 3$, be the automorphisms of S in Proposition 2.7 and define $\psi_j := \iota^{-1}\phi_j\iota : A_1 \rightarrow A_1$. The action of ψ_j on A_1 is given by the following table:

	y_0	y_1	y_2	y_3
ψ_1	$ibcy_1$	$-y_0$	$-by_3$	$-icy_2$
ψ_2	$iacy_2$	$ia y_3$	$-y_0$	cy_1
ψ_3	aby_3	$-ia y_2$	$-by_1$	$-iy_0$

A linear automorphism $\psi : A_1 \rightarrow A_1$ induces a linear automorphism of $\mathbb{P}(A_1^*)$ that sends the plane $y = 0$ to the plane $\psi(y) = 0$, and so on. For example, ψ_3 sends the plane $\varepsilon y_0 + \lambda y_1 + \mu y_2 + \nu y_3 = 0$ to the plane $\varepsilon aby_3 - \lambda ia y_2 - \mu by_1 - i\nu y_0 = 0$.

Let L be a commuting line defined by $\varepsilon y_0 + \lambda y_1 + \mu y_2 + \nu y_3 = \varepsilon' y_0 + \lambda' y_1 + \mu' y_2 + \nu' y_3 = 0$. The Plücker coordinates of L are given by the 2×2 minors of

$$\begin{pmatrix} \varepsilon & \lambda & \mu & \nu \\ \varepsilon' & \lambda' & \mu' & \nu' \end{pmatrix}.$$

Since $\psi_3(L)$ is given by the equations $\psi_3(\varepsilon y_0 + \lambda y_1 + \mu y_2 + \nu y_3) = \psi_3(\varepsilon' y_0 + \lambda' y_1 + \mu' y_2 + \nu' y_3) = 0$, the Plücker coordinates of $\psi_3(L)$ are given by the 2×2 minors of

$$\begin{pmatrix} -i\nu & -b\mu & -ia\lambda & ab\varepsilon \\ -i\nu' & -b\mu' & -ia\lambda' & ab\varepsilon' \end{pmatrix}.$$

Therefore

$$\begin{aligned}(z_{23} - z_{01})(\psi_3(L)) &= -iab(\lambda\varepsilon' - \lambda'\varepsilon) - ib(\nu\mu' - \nu'\mu) = ib(\alpha z_{01} + z_{23})(L) = 0, \\(z_{13} - z_{02})(\psi_3(L)) &= -a\beta(\mu\varepsilon' - \mu'\varepsilon) + a(\nu\lambda' - \nu'\lambda) = a(\beta z_{02} - z_{13})(L) = 0, \\(z_{12} - \gamma z_{03})(\psi_3(L)) &= iab(\mu\lambda' - \mu'\lambda) + iab\gamma(\nu\varepsilon' - \nu'\varepsilon) = iab(-z_{12} - \gamma z_{03})(L) = 0.\end{aligned}$$

In conclusion, the point on $\mathbb{G}(1, 3)$ corresponding to the line $\psi_3(L) = 0$ lies on the plane $z_{23} - z_{01} = z_{13} - z_{02} = z_{12} - \gamma z_{03} = 0$.

Similarly, the Plücker coordinates of $\psi_2(L)$ are given by the minors of

$$\begin{pmatrix} -\mu & c\nu & iac\varepsilon & ia\lambda \\ -\mu' & c\nu' & iac\varepsilon' & ia\lambda' \end{pmatrix}.$$

Therefore

$$\begin{aligned}(z_{23} + z_{01})(\psi_2(L)) &= -ac(\varepsilon\lambda' - \lambda\varepsilon') - c(\mu\nu' - \mu'\nu) = c(-\alpha z_{01} - z_{23})(L) = 0, \\(z_{13} + \beta z_{02})(\psi_2(L)) &= iac(\nu\lambda' - \nu'\lambda) - iac\beta(\mu\varepsilon' - \mu'\varepsilon) = iac(-z_{13} + \beta z_{02})(L) = 0, \\(z_{12} - z_{03})(\psi_2(L)) &= ia\gamma(\nu\varepsilon' - \nu'\varepsilon) + ia(\mu\lambda' - \mu'\lambda) = ia(-\gamma z_{03} - z_{12})(L) = 0.\end{aligned}$$

Thus, the point on $\mathbb{G}(1, 3)$ corresponding to $\psi_1(L) = 0$ lies on the plane $z_{23} + z_{01} = z_{13} + \beta z_{02} = z_{12} - z_{03} = 0$.

The Plücker coordinates of $\psi_1(L)$ are given by the minors of

$$\begin{pmatrix} -\lambda & ibc\varepsilon & -ic\nu & -b\mu \\ -\lambda' & ibc\varepsilon' & -ic\nu' & -b\mu' \end{pmatrix}.$$

Therefore

$$\begin{aligned}(z_{23} - \alpha z_{01})(\psi_1(L)) &= ibc(\nu\mu' - \nu'\mu) + ibc\alpha(\lambda\varepsilon' - \lambda'\varepsilon) = ibc(-z_{23} - \alpha z_{01})(L) = 0, \\(z_{13} + z_{02})(\psi_1(L)) &= -i\beta c(\varepsilon\mu' - \varepsilon'\mu) + ic(\lambda\nu' - \lambda'\nu) = ic(-\beta z_{02} + z_{13})(L) = 0, \\(z_{12} + z_{03})(\psi_1(L)) &= b\gamma(\varepsilon\nu' - \varepsilon'\nu) + b(\lambda\mu' - \lambda'\mu) = b(\gamma z_{03} + z_{12})(L) = 0.\end{aligned}$$

Thus, the point on $\mathbb{G}(1, 3)$ corresponding to $\psi_2(L) = 0$ lies on the plane $z_{23} - \alpha z_{01} = z_{13} + z_{02} = z_{12} + z_{03} = 0$.

To see that the conics are pairwise disjoint consider, first, $C_0 \cap C_1$. Since $z_{23} + \alpha z_{01}$ and $z_{23} - \alpha z_{01}$ vanish on $C_0 \cap C_1$, so do z_{01} and z_{23} . Since $z_{13} - \beta z_{02}$ and $z_{13} + z_{02}$ vanish on $C_0 \cap C_1$ so do z_{13} and z_{02} . Since $z_{12} + \gamma z_{03}$ and $z_{12} + z_{03}$ vanish on $C_0 \cap C_1$ so do z_{12} and z_{03} . Thus all the Plücker coordinates vanish on $C_0 \cap C_1$. Hence $C_0 \cap C_1 = \emptyset$.

The other cases are similar. ■

In the previous proof, the ψ_j 's act on the Plücker coordinates as

	z_{01}	z_{02}	z_{03}	z_{12}	z_{13}	z_{23}
ψ_1	$ibcz_{01}$	icz_{13}	bz_{12}	$b\gamma z_{03}$	$-i\beta cz_{02}$	$-ibcz_{23}$
ψ_2	$-cz_{23}$	$iacz_{02}$	iaz_{12}	$-ia\gamma z_{03}$	$-iacz_{13}$	$-\alpha cz_{01}$
ψ_3	$-ibz_{23}$	az_{13}	$iabz_{03}$	$-iabz_{12}$	$a\beta z_{02}$	$iabz_{01}$

7. LINE MODULES FOR A

7.1. Classification of line modules. Let $\mathbb{L} \subset \mathbb{G}(1, 3) \subseteq \mathbb{P}(\wedge^2 A_1^*) \cong \mathbb{P}^5$ denote the line scheme for A .

Theorem 7.1. *Let ξ_1, ξ_2, ξ_3 be the 2-torsion points of E . The line scheme for A is a reduced and irreducible curve of degree 20. It is the union of 3 disjoint quartic elliptic curves and 4 disjoint plane conics,*

$$\mathbb{L} = \left((E/\langle \xi_1 \rangle) \sqcup (E/\langle \xi_2 \rangle) \sqcup (E/\langle \xi_3 \rangle) \right) \cup \left(C_0 \sqcup C_1 \sqcup C_2 \sqcup C_3 \right)$$

having the property that $|(E/\langle \xi_i \rangle) \cap C_j| = 2$ for all $(i, j) \in \{1, 2, 3\} \times \{0, 1, 2, 3\}$.

Proof. We consider $\mathbb{G}(1, 3)$ as a subvariety of \mathbb{P}^5 via the Plücker embedding. By [27, Cor. 2.6], every component of \mathbb{L} has dimension ≥ 1 . By Theorem A.1, $\dim(\mathbb{L}) \leq 1$ so every component of \mathbb{L} has dimension $= 1$. Hence, by [10], $\deg(\mathbb{L}) = 20$.

By [9, Prop. 11.12], the image of each $E/\langle \xi \rangle$ in $\mathbb{G}(1, 3)$ has degree 4 as a curve in \mathbb{P}^5 . Each of the conics C_0, C_1, C_2, C_3 has degree two. Thus, the union of these 7 components of \mathbb{L} has degree $3 \times 4 + 4 \times 2 = 20$. It follows that \mathbb{L} is as claimed.

The claim about the intersection points of the components of \mathbb{L} is proved in Corollary 11.4. It is also a consequence of the calculations in the Appendix. ■

Proposition 7.2. *For $j = 1, 2, 3$, the image of $E/\langle \xi_j \rangle$ in $\mathbb{G}(1, 3)$ is the curve E_j described in the Appendix. Defining equations for E_j are given at the end of the Appendix and the points in $C_i \cap E_j$ are given in Table 6.*

Proof. We just do the case $j = 1$. The other cases are similar. By [9, Lem. 11.7], a secant line of the form $\overline{p, p + \xi_1}$ is cut out by equations of the form $\beta_0 y_0 + \beta_1 y_1 = \beta_2 y_2 + \beta_3 y_3 = 0$. It is easy to see that the point in $\mathbb{G}(1, 3)$ corresponding to this line lies on the 3-plane $z_{01} = z_{23} = 0$. But the only component of \mathbb{L} contained in that 3-plane is the curve E_1 in the Appendix. ■

7.2. We now give a more detailed description of the action of $E[4]$ on the elliptic and conic families of lines induced by the action on $\text{Gr}(A)$ discussed above.

Proposition 7.3. *The group $E[4] \cong (\mathbb{Z}/4)^2$ acts on the three elliptic curves and the four conics that parametrize the line modules as follows:*

- (1) $E[4]$ fixes each elliptic family individually and acts on the family $E/\langle \xi \rangle$, $\xi \in E[2]$, as translations by the image of $E[4]$ through $E \rightarrow E/\langle \xi \rangle$.
- (2) The subgroup $\Gamma \cong E[2] \subset E[4]$ fixes each of the conics $\{C_i \mid 0 \leq i \leq 3\}$ individually and acts on the commuting conic C_0 by twisting by the algebra automorphisms

$$(7-1) \quad y_0 \mapsto y_0, \quad y_i \mapsto y_i, \quad y_j \mapsto -y_j, \quad y_k \mapsto -y_k$$

for the three cyclic permutations (i, j, k) of $(1, 2, 3)$.

The quotient group $E[4]/E[2] \cong (\mathbb{Z}/2)^2$ acts on the set $\{C_i \mid 0 \leq i \leq 3\}$ as the regular permutation representation.

Proof. (1) Fix some $\xi \in E[2]$. Let $g : \mathbb{P}(S_1^*) \rightarrow \mathbb{P}(A_1^*)$ be the identification made in §3.2. By [9, Proposition 10.10], the closed immersion of $E/\langle \xi \rangle$ into the line scheme for A is given by

$$(7-2) \quad p + \langle \xi \rangle \mapsto g(\text{the line in } \mathbb{P}(S_1^*) \text{ passing through } p \text{ and } p + \xi).$$

Translation by the image of $\varepsilon_1 \in E[4]$ in $E/\langle \xi \rangle$ moves the line through p and $p + \xi$ to that through $p + \varepsilon_1$ and $p + \varepsilon_1 + \xi$. Let $U \subseteq S_1$ be the two-dimensional subspace annihilating a generator of the line module $M_{p, p + \xi}$, and hence the points p and $p + \xi$. Applying the automorphism $\varepsilon_1 \in H_4 \subseteq \text{Aut}(S)$ to U , $\varepsilon_1 U$ annihilates a generator of the new line module.

The action of H_4 on A_1 is defined so that the identification in §3.2, and hence the map g , is H_4 -equivariant. Hence, transporting the ε_1 -translation to the right hand side of (7-2), we see that the new line module obtained in this fashion is again annihilated by $\varepsilon_1 U$. This, however, matches the description of the action of the autoequivalence a_{ε_1} on graded A -modules given in §6.7:

namely, for any such module M , $a_{\varepsilon_1} M$ coincides with M as a graded vector space and the action of the degree-one elements of A on even-degree elements of M is twisted by ε_1^{-1} . Hence, if $U \subset A_1$ annihilates the degree-zero component of M , then $\varepsilon_1 U$ will annihilate the degree-zero component of $a_{\varepsilon_1} M$.

(2) The fact that two generators of $E[4] \cong (\mathbb{Z}/4)^2$ (a_{ε_1} and a_{ε_2} say) act on the four-element set $\{C_i\}$ as products of two involutions follows from Corollary 6.8. In particular, the 2-torsion $E[2]$ of $E[4]$ preserves each conic C_i .

On the other hand, by Lemma 6.3, the group algebra of the subgroup $\langle \varepsilon_1^2, \varepsilon_2^2, \delta \rangle \subseteq H_4$ retains its coalgebra structure after the comultiplication of kH_4 has been deformed into that of H^* . Therefore $\langle \varepsilon_1^2, \varepsilon_2^2, \delta \rangle$, and in particular Γ , acts on A by graded algebra automorphisms. Since the identification of S_1 and A_1 is of the form $x_i \leftrightarrow$ a scalar multiple of y_i , this Γ -action is precisely (7-1). ■

7.3. Elliptic line modules. The following is an easy consequence of Proposition 2.4.

Proposition 7.4. *Let ξ , ξ' , and $\xi + \xi'$, be the 2-torsion points in E . Let $p, z \in E$. Define $x = p + E[2]$. The following statements are equivalent:*

- (1) $\Omega(z)$ annihilates $M_{p,p+\xi}$;
- (2) $\Omega(z)$ annihilates $M_{p+\xi',p+\xi'+\xi}$;
- (3) $\Omega(z)$ annihilates $M_{x,\xi}$, the module defined in (3-3);
- (4) $2p + \xi \in \{z, -z - 2\tau\}$;
- (5) $z \in \{2p + \xi, -2p - 2\tau + \xi\}$.

Proposition 7.5. *Let $z \in E$. Then $\Omega(z)$ annihilates*

- (1) *exactly six modules of the form $M_{x,\xi}$ if $z + \tau \notin E[2]$, and*
- (2) *exactly three modules of the form $M_{x,\xi}$ if $z + \tau \in E[2]$.*

Proof. Let $A(z) = \{(x, \xi) \mid \Omega(z)M_{x,\xi} = 0\}$ and $S(z) = \{(p + E[2], \xi) \mid 2p + \xi = z\}$. By Corollary 7.4, $A(z) = S(z) \cup S(-z - 2\tau)$.

For each of the three 2-torsion points ξ there is a unique coset $p + E[2]$ such that $2p + \xi = z$. Hence $|S(z)| = 3$. If $z + \tau \notin E[2]$, then $z \neq -z - 2\tau$ so $S(z) \cap S(-z - 2\tau) = \emptyset$, whence $|A(z)| = 6$. If $z + \tau \in E[2]$, then $z = -z - 2\tau$ so $S(z) = S(-z - 2\tau)$, whence $|A(z)| = 3$. ■

Proposition 7.6. *Let $z \in E$. Then $\Omega(z)$ annihilates*

- (1) *exactly four elliptic line modules in each elliptic family if $z + \tau \notin E[2]$, and*
- (2) *exactly two elliptic line modules in each elliptic family if $z + \tau \in E[2]$.*

Let ξ and ξ' be different 2-torsion points. If p is one of the four points on E such that $2p = z + \xi$, then the line modules parametrized by $E/\langle \xi \rangle$ that are annihilated by $\Omega(z)$ correspond to the points on $E/\langle \xi \rangle$ that are the images of p , $p + \xi'$, $-p - \tau$, and $-p - \tau + \xi'$.

7.4. By Proposition 2.19, $\text{Hom}_{\text{Gr}(S)}(M_{p,q}, \tilde{V}(\tau + \xi)) \neq 0$ for all $p + q = \tau + \xi$.

Let $F_{\tau+\xi} = \tilde{V}(\tau + \xi) \otimes k^2 \in \text{Gr}(S')$.

Lemma 7.7. *Let $p \in E$ and write $x = p + E[2]$. The following conditions are equivalent:*

- (1) $\text{Hom}_{\text{Gr}(S)}(M_{p,p+\xi}, \tilde{V}(\tau + \xi)) \neq 0$;
- (2) $\text{Hom}_{\text{Gr}(S)}(M_{p+\xi',p+\xi'+\xi}, \tilde{V}(\tau + \xi)) \neq 0$;
- (3) $\text{Hom}_{\text{Gr}(S')}(M_{x,\xi}, F_{\tau+\xi}) \neq 0$.

Proof. Because $p + p + \xi = (p + \xi') + (p + \xi' + \xi)$ this follows from Proposition 2.19 and the Morita equivalence between S and S' . ■

Proposition 7.8. *Fix $\xi \in E[2]$. For each 2-torsion point ω , there is a unique $x \in E/E[2]$ such that $\text{Hom}_{\text{Gr}(S')}(M_{x,\omega}, F_{\tau+\xi}) \neq 0$, namely $x = p + E[2]$ where $2p = \tau + \xi + \omega$.*

Proof. The points $p \in E$ such that $2p = \tau + \xi + \omega$ form a single $E[2]$ -coset; hence x does not depend on the choice of p . Since $2p + \omega = \tau + \xi$, $\text{Hom}_{\text{Gr}(S)}(M_{p,p+\omega}, \tilde{V}(\tau + \xi)) \neq 0$ by Proposition 2.19. Hence $\text{Hom}_{\text{Gr}(S')}(M_{x,\omega}, F_{\tau+\xi}) \neq 0$ by Lemma 7.7.

Let $y = q + E[2]$. If $\text{Hom}_{\text{Gr}(S')}(M_{y,\omega}, F_{\tau+\xi}) \neq 0$, then $\text{Hom}_{\text{Gr}(S)}(M_{q,q+\omega}, \tilde{V}(\tau + \xi)) \neq 0$ by Lemma 7.7 so $2q + \omega = \tau + \xi = 2p + \omega$. Hence $p - q \in E[2]$ and $x = y$. ■

Let $p \in \mathfrak{P}_\infty$ and let $M_p \in \text{Gr}(A)$ be the corresponding point module. Since three of the coordinate functions y_0, y_1, y_2, y_3 vanish at p , M_p is annihilated by $\Omega(\xi)$ for some $\xi \in E[2]$. Since $\tau \notin E[2]$, there are, up to isomorphism, exactly twelve line modules in $\text{Gr}(A)$ that are annihilated by $\Omega(\xi)$.

8. POINTS ON CONIC LINES

8.1. This section answers the following question: if $p \in \mathfrak{P}$, i.e., $M_p = A/Ap^\perp$ is a point module in $\text{Gr}(A)$, and $L = A/Al^\perp$ is a conic line module, when is there an epimorphism $\pi^*L \rightarrow \pi^*M_p$ in $\text{QGr}(A)$. Since M_p is 1-critical, this is equivalent to the question “if $p \in \mathfrak{P}$ and ℓ is a conic line in $\mathbb{P}(A_1^*)$ when is there a non-zero homomorphism $A/Al^\perp \rightarrow A/Ap^\perp$?” Clearly, there is a non-zero homomorphism $A/Al^\perp \rightarrow A/Ap^\perp$ if and only if $p \in \ell$ so the original question about morphisms in $\text{Proj}_{nc}(A)$ is equivalent to the geometric question “if $p \in \mathfrak{P}$ which conic lines does p lie on?”

We use the explicit descriptions of \mathfrak{P} and the conic line modules to answer the question.

Theorem 8.1. *The lines in $\text{Proj}_{nc}(A)$, which are also lines in $\mathbb{P}(A_1^*)$, parametrized by the conics C_0, \dots, C_3 , are such that*

- (1) *every point in $\mathfrak{P} - (\mathfrak{P}_\infty \cup \mathfrak{P}_j)$ lies on exactly one line from C_j and*
- (2) *no point in $\mathfrak{P}_\infty \cup \mathfrak{P}_j$ lies on any lines from C_j .*

If $0 \leq i \neq j \leq 3$, the lines in C_i that pass through a point in \mathfrak{P}_j form a single Γ -orbit in the sense of Proposition 7.3.

Proof. Since C_j is the image of C_0 under the autoequivalences a_{ε_j} , both (1) and (2) follow from Propositions 5.5 and 6.11 together with the fact that the $E[4]$ -action on the point scheme permutes the ordinary points, and therefore permutes the special points too. The last sentence follows from the fact that the points in \mathfrak{P}_j form a single Γ -orbit. ■

The obvious way to get a family of lines in \mathbb{P}^3 parametrized by a smooth conic is to take the lines in a ruling on a smooth quadric. We now determine quadrics Q_j such that the lines parametrized by C_j belong to a ruling on Q_j . The following result is a standard exercise.

Proposition 8.2. *Let $\lambda, \mu, \nu \in k^\times$ and $(s, t) \in \mathbb{P}^1$. Let Q be the quadric $y_0^2 - \lambda^2 y_1^2 + \mu^2 y_2^2 - \nu^2 y_3^2 = 0$. The line*

$$(8-1) \quad s(y_0 - \lambda y_1) + t(\mu y_2 - \nu y_3) = t(y_0 + \lambda y_1) - s(\mu y_2 + \nu y_3) = 0$$

lies on Q and the corresponding point on $\mathbb{G}(1, 3)$ is

$$(8-2) \quad (2\lambda st, -\mu(s^2 + t^2), \nu(t^2 - s^2), \lambda\mu(s^2 - t^2), \lambda\nu(s^2 + t^2), -2\mu\nu st)$$

which lies on the intersection of $\mathbb{G}(1, 3)$ with the plane

$$(8-3) \quad z_{23} + \lambda^{-1}\mu\nu z_{01} = z_{13} + \mu^{-1}\lambda\nu z_{02} = z_{12} + \nu^{-1}\lambda\mu z_{03} = 0.$$

The lines in the other ruling on Q correspond to the points on the intersection of $\mathbb{G}(1, 3)$ with the plane

$$(8-4) \quad z_{23} - \lambda^{-1}\mu\nu z_{01} = z_{13} - \mu^{-1}\lambda\nu z_{02} = z_{12} - \nu^{-1}\lambda\mu z_{03} = 0.$$

Proof. If we write the equation defining Q as $(y_0 + \lambda y_1)(y_0 - \lambda y_1) = (\nu y_3 + \mu y_2)(\nu y_3 - \mu y_2)$ it is easy to see that the line (8-1) lies on Q . The Plücker coordinates of the corresponding point on $\mathbb{G}(1, 3)$ are given by the minors of the matrix

$$\begin{pmatrix} s & -s\lambda & t\mu & -t\nu \\ t & t\lambda & -s\mu & -s\nu \end{pmatrix}.$$

Those minors are the entries in (8-2). The point in (8-2) obviously lies on the plane in (8-3).

Replacing (λ, μ, ν) by $(-\lambda, -\mu, -\nu)$ gives the other ruling on Q and (8-3) changes to (8-4). ■

Corollary 8.3. *The conics C_0, \dots, C_3 , are the intersections of $\mathbb{G}(1, 3)$ with the planes in $\mathbb{P}(\wedge^2 A_1^*)$ in the second column of Table 5, and the lines in $\mathbb{P}(A_1^*)$ that correspond to the points on C_j provide a ruling on the quadric Q_j in the third column of the following table:*

	$C_j = \mathbb{G}(1, 3) \cap (\text{the plane below})$	The quadric Q_j
C_0	$z_{23} + \alpha z_{01} = z_{13} - \beta z_{02} = z_{12} + \gamma z_{03} = 0$	$y_0^2 + \beta \gamma y_1^2 + \gamma \alpha y_2^2 + \alpha \beta y_3^2 = 0$
C_1	$z_{23} - \alpha z_{01} = z_{13} + z_{02} = z_{12} + z_{03} = 0$	$y_0^2 - y_1^2 - \alpha y_2^2 + \alpha y_3^2 = 0$
C_2	$z_{23} + z_{01} = z_{13} + \beta z_{02} = z_{12} - z_{03} = 0$	$y_0^2 + \beta y_1^2 - y_2^2 - \beta y_3^2 = 0$
C_3	$z_{23} - z_{01} = z_{13} - z_{02} = z_{12} - \gamma z_{03} = 0$	$y_0^2 - \gamma y_1^2 + \gamma y_2^2 - y_3^2 = 0$

TABLE 5. The quadric ruled by the lines in C_j .

Further, $\mathfrak{P} - (\mathfrak{P}_\infty \cup \mathfrak{P}_j) \subseteq Q_j$ and each point in $\mathfrak{P} - (\mathfrak{P}_\infty \cup \mathfrak{P}_j)$ lies on a unique line belonging to C_j .

Proof. Apply Proposition 8.2 with (λ, μ, ν) equal to $(-ibc, ac, iab)$, $(1, ia, ia)$, (ib, i, b) , and $(c, c, -1)$.

One sees that $\mathfrak{P} - (\mathfrak{P}_\infty \cup \mathfrak{P}_j) \subseteq Q_j$ by evaluating the equation for Q_j at the points in $\mathfrak{P} - (\mathfrak{P}_\infty \cup \mathfrak{P}_j)$. The calculation is simplified by noticing that the values of $(y_0^2, y_1^2, y_2^2, y_3^2)$ on $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$, and \mathfrak{P}_3 , are $(1, 1, 1, 1)$, $(\beta\gamma, -1, -\beta, \gamma)$, $(\alpha\gamma, \alpha, -1, -\gamma)$, and $(\alpha\beta, -\alpha, \beta, -1)$, respectively. Finally, since Q_j is the disjoint union of the lines in the ruling on it parametrized by C_j , each point in $\mathfrak{P} - (\mathfrak{P}_\infty \cup \mathfrak{P}_j)$ lies on a unique line belonging to C_j . ■

8.1.1. *Remark.* If we identify $\mathbb{P}(A_1^*)$ with $\mathbb{P}(S_1^*)$ according to the convention in §3.2 the quadrics Q_j become quadrics in $\mathbb{P}(S_1^*)$. For example, in the x_0, x_1, x_2, x_3 coordinates, Q_0 is the zero locus of $x_0^2 - \beta\gamma x_1^2 - \gamma\alpha x_2^2 + \alpha\beta x_3^2$. It is reasonable to ask if these quadrics contain E . They do not. For example, Q_0 does not contain the point $\tau' + \varepsilon_1 = (a, -ia, i, 1)$.

9. POINTS ON ELLIPTIC LINES

In this section we determine which point modules are quotients of which elliptic line modules.

9.1. Let $\xi \in E[2]$. As in §7.4 we write $F_{\tau+\xi}$ for $\tilde{V}(\tau + \xi) \otimes k^2$. By Proposition 7.8, there are at most three modules $M_{x,\omega} \in \text{Gr}(S')$ such that $\text{Hom}_{\text{Gr}(S')}(M_{x,\omega}, F_{\tau+\xi}) \neq 0$.

Lemma 9.1. *Fix a Γ -equivariant structure on $F_{\tau+\xi} \in \text{Gr}(S')$ and a non-zero morphism $f : M_{x,\omega} \rightarrow F_{\tau+\xi}$ in $\text{Gr}(S')$. There is at most one Γ -equivariant structure on $M_{x,\omega}$ for which f is Γ -equivariant.*

Proof. Let $f : M_{x,\omega} \rightarrow F_{\tau+\xi}$ be a non-zero morphism in $\text{Gr}(S')$ that is Γ -equivariant for some Γ -equivariant structure on $M_{x,\omega}$.

Suppose $x = p + E[2]$ and let $\{\omega, \omega', \omega''\}$ be the 2-torsion points on E . By Proposition 2.19 and the equivalence $\text{Gr}(S) \cong \text{Gr}(S')$, there are, up to scaling, unique non-zero morphisms $M_{p,p+\omega} \otimes k^2 \rightarrow$

$F_{\tau+\xi}$ and $M_{p+\omega', p+\omega''} \otimes k^2 \rightarrow F_{\tau+\xi}$ in $\text{Gr}(S')$. If these homomorphisms are the restrictions of f then they are non-zero because f is Γ -equivariant. Therefore f restricts to non-zero $\langle \omega \rangle$ -equivariant maps $M_{p, p+\omega} \otimes k^2 \rightarrow F_{\tau+\xi}$ and $M_{p+\omega', p+\omega''} \otimes k^2 \rightarrow F_{\tau+\xi}$. The ideas in the proof of [Proposition 2.17\(2\)](#) show that $\tilde{V}(\tau + \xi)_0$ is the direct sum of the images of the degree zero components of the non-zero maps $M_{p, p+\omega} \rightarrow \tilde{V}(\tau + \xi)$ and $M_{p+\omega', p+\omega''} \rightarrow \tilde{V}(\tau + \xi)$.

But f commutes with the action of ω , so it maps the $(+1)$ - and (-1) -eigenspaces for the action of ω on the degree-zero component of $M_{x, \omega} \otimes k^2$ isomorphically to the $(+1)$ - and (-1) -eigenspaces for the action of ω on $(F_{\tau+\xi})_0$. This uniquely determines the Γ -equivariant structure on $M_{x, \omega}$ because, by [9, Thm. 11.6(3)], interchanging the $(+1)$ - and (-1) -eigenspaces for ω on $M_{x, \omega}$ switches between the two equivariant structures. ■

[Lemma 9.1](#) puts an upper bound on the number of elliptic line modules in $\text{Gr}(A)$ that map onto a given ordinary point module. We will soon see that this bound is achieved.

Moreover, the situation is different for the four special point modules. If $P \in \text{Gr}(S)$ is a special point module, then $(P^{\oplus 2}) \otimes k^2$ is a quotient of exactly three S' -modules of the form $M_{x, \xi}$. Indeed, this is a simple count once we recall that P is a quotient of $M_{p, q}$ precisely when $p + q$ is the element of $E[2]$ associated to P .

Theorem 9.2. *Let $\xi \in E[2] - \{o\}$ and let $p \in \mathfrak{P}$.*

- (1) *If $p \in \mathfrak{P} - \mathfrak{P}_\infty$, then p lies on exactly one line in the family parametrized by $E/\langle \xi \rangle$.*
- (2) *If $p \in \mathfrak{P}_\infty$, then p lies on exactly two lines in the family parametrized by $E/\langle \xi \rangle$.*

Proof. Let $P \in \text{Gr}(S)$ be the point module corresponding to p .

(1) If $p \in \mathfrak{P}_j$, let $F = \tilde{V}(\tau + \xi_j) \otimes k^2$, and give F the unique equivariant S' -module structure such that $P \cong F^\Gamma$.

By [Proposition 7.8](#), there is a unique $x \in E[2]$ such that $\text{Hom}_{\text{Gr}(S')}(M_{x, \xi}, F) \neq 0$. Fix that x and fix $q \in E$ such that $x = q + E[2]$. Let f be a non-zero map in $\text{Hom}_{\text{Gr}(S')}(M_{x, \xi}, F)$. By [Lemma 9.1](#), there is at most one equivariant structure on $M_{x, \xi}$ for which f is Γ -equivariant. We give $M_{x, \xi}$ that equivariant structure.

Given that equivariant structure, every non-zero graded S' -module homomorphism from the summand $M_{q, q+\xi} \otimes k^2$ of $M_{x, \xi}$ to F is equivariant for the ξ -action in degree zero. That it is equivariant in all degrees then follows from the fact that $M_{q, q+\xi} \otimes k^2$ is a cyclic S' -module because every homogeneous element in $M_{q, q+\xi} \otimes k^2$ can be obtained by acting on a degree-zero element by an eigenvector of ξ .

A non-zero graded S' -module map from the other summand $M_{q+\xi', q+\xi''}$ of $M_{x, \xi}$ to F can then be chosen uniquely so that the resulting map

$$M_{x, \xi} = (M_{q, q+\xi} \oplus M_{q+\xi', q+\xi''}) \otimes k^2 \rightarrow F$$

intertwines the Γ -action at the degree-zero level. Once more it will then be equivariant in all degrees, since higher-degree homogeneous elements can be obtained by acting on degree-zero elements with eigenvectors of Γ in S' and the map is an S' -module map.

(2) Fix equivariant structures on $M_{x, \xi}$ and $(P^{\oplus 2}) \otimes k^2$. The degree-zero component of each module is isomorphic as a Γ -module to the regular representation of Γ , so there is a unique way to match up the respective eigenspaces. Moreover, there is, up to scaling, a unique non-zero morphism $M_{x, \xi} \rightarrow (P^{\oplus 2}) \otimes k^2$ in $\text{Gr}(S')$ that agrees with this matching. This morphism must then be Γ -equivariant as in the proof of part (1). ■

10. FAT POINTS ON ELLIPTIC LINES

This section determines which of the fat point modules $M_p \otimes k^2$ described in §3.4 lie on which elliptic lines. Thus we answer the following question: if L is an elliptic line module, when is $\text{Hom}_{\text{Gr}(A)}(L, M_p \otimes k^2) \neq 0$. If $f : L \rightarrow M_p \otimes k^2$ is a non-zero morphism in $\text{Gr}(A)$, then f becomes

an epimorphism in $\mathbf{QGr}(A)$. Recall that $M_p \otimes k^2$ is a simple object in $\mathbf{QGr}(A)$ and corresponds to the skyscraper sheaf $\mathcal{O}_{p+E[2]}$ under the equivalence $\mathbf{QGr}(\tilde{B}) \equiv \mathbf{Qcoh}(E/E[2])$.

10.1. Elliptic line modules are obtained by descent from the Γ -equivariant structures on the S' -modules $M_{x,\xi}$. To understand their relation to the modules $M_p \otimes k^2$ we need an equivariant description of $M_p \otimes k^2$ too.

The A -module $M_p \otimes k^2$ is isomorphic to N^Γ for some Γ -equivariant S' -module $N = N' \otimes k^2$. Since $\omega^* M_p \cong M_{p+\omega}$ for all $\omega \in \Gamma \equiv E[2]$, the obvious candidate for N is the module in (10-1).

Lemma 10.1. *Let $p \in E$. There is a unique Γ -equivariant structure on the S' -module*

$$(10-1) \quad N = \bigoplus_{\omega \in E[2]} M_{p+\omega} \otimes k^2$$

and $N^\Gamma \cong M_p \otimes k^2$ as A -modules.

Proof. If $\omega \in E[2]$, then $\omega^* M_p \cong M_{p+\omega}$. Hence $\omega^* N \cong N$. Since the S -modules M_p and M_q are isomorphic if and only if $p = q$, $\text{Aut}_{S'}(N)$ is isomorphic to $(k^\times)^4$. We label the elements of $\text{Aut}_{S'}(N)$ as 4-tuples $\lambda = (\lambda_\omega)_{\omega \in E[2]}$ with the convention that λ acts on the summand $M_{p+\omega} \otimes k^2$ of N as multiplication by λ_ω . Arguing as in [9, Lem. 10.3], $\text{Aut}_{S'}(N)$ is a Γ -module with $\xi \in \Gamma$ acting as follows:

$$\xi \triangleright (\lambda_\omega)_{\omega \in E[2]} = (\mu_\omega)_{\omega \in E[2]}$$

where $\mu_\omega = \lambda_{\omega+\xi}$.

To prove the existence and uniqueness of a Γ -equivariant structure on N we argue as in [9, Prop. 10.5]. Thus, it suffices to show that

$$(10-2) \quad H^1(\Gamma, \text{Aut}_{S'}(N)) = H^2(\Gamma, \text{Aut}_{S'}(N)) = 0$$

because the vanishing of H^2 implies the existence of at least one equivariant structure and, since H^1 acts simply transitively on the set of equivariant structures, the vanishing of H^1 implies there is at most one equivariant structure.

To check (10-2) note that $\text{Aut}_{S'}(N)$ can be identified with the group of k^\times -valued functions on Γ , acted upon by Γ via translation on the domain of the functions. In other words, there is an isomorphism of $k\Gamma$ -modules

$$\text{Aut}_{S'}(N) \cong \text{Hom}_{\mathbb{Z}}(k\Gamma, (k^\times)^4)$$

where, on the right, the \mathbb{Z} -action is additive on $k\Gamma$ and multiplicative on $(k^\times)^4$. This is sometimes called a *relatively injective* Γ -module, and its higher cohomology vanishes [25, Prop. VII.2.1].

The summands $M_{p+\omega} \otimes k^2$, $\omega \in E[2]$, are isomorphic to each other so the invariant part $N^\Gamma \in \mathbf{Gr}(A)$ is isomorphic to $M_p \otimes k^2$. ■

Let $\xi \in E[2] - \{o\}$ and let $x \in E/\langle \xi \rangle$. The space $\text{Hom}_{\mathbf{Gr}(S')}(M_{x,\xi}, M_p \otimes k^2)$ is non-zero if and only if the image of x in $E/E[2]$ equals $p + E[2]$ in which case it is isomorphic to k^2 . Hence $\text{Hom}_{\mathbf{Gr}(S')}(M_{x,\xi}, N) \cong k^8$. We would like some of these maps to be equivariant for at least one of the two Γ -equivariant structures on $M_{x,\xi}$, in order to have them descend to A . The next result shows this is the case.

Proposition 10.2. *If $p \in E$, then $M_p \otimes k^2$ is a quotient in $\mathbf{QGr}(A)$ of exactly two line modules in each of the three elliptic families.*

Proof. Let $x = p + E[2]$ and fix $\xi \in E[2]$. Let N be the Γ -equivariant S' -module in (10-1) equipped with the Γ -module structure in Lemma 10.1. We must show that for each of the two Γ -equivariant structures on $M_{x,\xi}$ there is a non-zero homomorphism $M_{x,\xi} \rightarrow N$ of Γ -equivariant S' -modules.

The actions of Γ on the Γ -equivariant S' -modules $M_{x,\xi}$ and N induce a Γ -module structure on $\text{Hom}_{\mathbf{Gr}(S')}(M_{x,\xi}, N)$. We must show that $\text{Hom}_{\mathbf{Gr}(S')}(M_{x,\xi}, N)^\Gamma \neq 0$.

Since $M_{x,\xi}$ is generated in degree zero, $\text{Hom}_{\text{Gr}(S')}(M_{x,\xi}, N)$ is isomorphic as a Γ -module to $\text{Hom}_{M_2(k)}((M_{x,\xi})_0, N_0)$. The fact that the trivial Γ -representation occurs in it follows from the observation that both $(M_{x,\xi})_0$ and N_0 are Γ -equivariant $M_2(k)$ -modules and, by descent, the category of Γ -equivariant $M_2(k)$ -modules is equivalent to Vect . ■

Using this, we can determine how the group $E[4]$ of autoequivalences of $\text{Gr}(A)$ acts on the modules $M_p \otimes k^2$. Let us first record the following observation.

Proposition 10.3. *The action of $E[4]$ on $\text{Gr}(A)$ introduced in §6.7 restricts to an action of $E[4]$ as auto-equivalences of $\text{Gr}(\tilde{B})$.*

Proof. By Proposition 6.5, the auto-equivalence a_{ε_j} sends modules annihilated by $\Omega(z)$ to modules annihilated by $\Omega(z + \xi_j)$. The result now follows from the fact that $\text{Gr}(\tilde{B})$ consists of the A -modules that are annihilated by all $\Omega(z)$, $z \in E$. ■

Since the modules $M_p \otimes k^2$ are, up to isomorphism, the only 1-critical \tilde{B} -modules with Hilbert series $2(1-t)^{-1}$, it follows from Proposition 10.3 that $E[4]$ acts on the elliptic curve that parametrizes them.

Proposition 10.4. *The action of $E[4]$ on the set of 1-critical \tilde{B} -modules with Hilbert series $2(1-t)^{-1}$ induces an action of $E[4]$ on $E/E[2]$ as translations by the image of $E[4]$ in $E/E[2]$.*

Proof. Fix $\xi \in E[2]$. By Proposition 10.2, $M_p \otimes k^2$ (whose isomorphism class only depends on $p + E[2]$) lies on both lines that comprise the preimage of $p + E[2]$ through $E/\langle \xi \rangle \rightarrow E/E[2]$ and only on those. By part (1) of Proposition 7.3, $E[4]$ acts as translations on the domain of this covering map, and hence it must act in the same fashion on its codomain $E/E[2]$. ■

11. FAT POINTS ON CONIC LINES

This section determines the incidence relations between the “lines” in $\text{Proj}_{nc}(A)$ parametrized by the conics C_0, C_1, C_2, C_3 , and the “points” on the curve $E/E[2] \subseteq \text{Proj}_{nc}(A)$.

11.1. Since \tilde{B} is a quotient of A , [33, Thm.1.2] shows that $\text{Proj}_{nc}(\tilde{B})$ is a closed subspace of $\text{Proj}_{nc}(A)$ in the sense of [44, §3.3] and [37, Defn. 2.4]. Since $\text{QGr}(\tilde{B}) \equiv \text{Qcoh}(E/E[2])$ we say that $\text{Proj}_{nc}(\tilde{B})$ is isomorphic to $E/E[2]$ and think of $E/E[2]$ as a closed curve in the ambient non-commutative variety $\text{Proj}_{nc}(A)$.

Lemma 11.1. *Fix a point $p = (\delta_0, \delta_1, \delta_2, \delta_3) \in E \subseteq \mathbb{P}(S_1^*)$. Let $f_p : A_1 \rightarrow M_2(k)$ be the linear map*

$$\begin{aligned} f_p(\lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3) &:= \begin{pmatrix} \lambda_0 \delta_0 + i \lambda_1 \delta_1 & i \lambda_2 \delta_2 - \lambda_3 \delta_3 \\ i \lambda_2 \delta_2 + \lambda_3 \delta_3 & \lambda_0 \delta_0 - i \lambda_1 \delta_1 \end{pmatrix} \\ &= \lambda_0 \delta_0 q_0 + \lambda_1 \delta_1 q_1 + \lambda_2 \delta_2 q_2 + \lambda_3 \delta_3 q_3. \end{aligned}$$

- (1) *Let $y \in A_1$. If $e \otimes v \in (M_p \otimes k^2)_0 - \{0\}$, then $y(e \otimes v) = 0$ if and only if $f_p(y)v = 0$.*
- (2) *f_p is a linear isomorphism if and only if $p \notin E[4]$.*
- (3) *Suppose $p \notin E[4]$. As kv varies over the points in $\mathbb{P}(k^2)$, f_p provides a bijection between the 2-dimensional subspaces $ky + ky'$ of A_1 such that $\text{Hom}_{\text{Gr}(A)}(A/Ay + Ay', M_p \otimes k^2) \neq 0$ and the simple left ideals of $M_2(k)$.*

Proof. (1) There is a basis e' for $(M_p)_1$ such that $x_j e = \delta_j e'$ for $j = 0, 1, 2, 3$. Since $\sum_{j=0}^3 \lambda_j y_j = \sum_{j=0}^3 \lambda_j x_j \otimes q_j$,

$$\left(\sum_{j=0}^3 \lambda_j y_j \right) (e \otimes v) = \sum_{j=0}^3 x_j e \otimes \lambda_j q_j v = e' \otimes f_p \left(\sum_{j=0}^3 \lambda_j y_j \right) v.$$

Thus, $y \in A_1$ annihilates $e \otimes v$ if and only if $f_p(y)$ annihilates v .

(2) Since $\{1, q_1, q_2, q_3\}$ is linearly independent, f_p is an isomorphism if and only if $\delta_0\delta_1\delta_2\delta_3 \neq 0$. By [Corollary 2.13](#), $\delta_0\delta_1\delta_2\delta_3 = 0$ if and only if $p \in E[4]$.

(3) The correspondence $kv \longleftrightarrow \text{Ann}_{M_2(k)}(v)$ is a bijection between the points in $\mathbb{P}(k^2) = \mathbb{P}^1$ and the simple left ideals of $M_2(k)$. The composed bijection $kv \longleftrightarrow \text{Ann}_{M_2(k)}(v) \xrightarrow{f_p} \text{Ann}_{A_1}(e \otimes v)$ is a bijection between the points in $\mathbb{P}(k^2) = \mathbb{P}^1$ and 2-dimensional subspaces $ky + ky'$ of A_1 that annihilate a non-zero element in $(M_p \otimes k^2)_0$. ■

We now apply [Lemma 11.1](#) with p equal to $\tau' := (abc, a, b, c)$. By [Proposition 2.14](#), $2\tau' = -\tau$.

Proposition 11.2. *Let $f = f_{\tau'} : A_1 \rightarrow M_2(k)$ be the linear isomorphism defined in [Lemma 11.1](#).*

- (1) *f gives a bijection between commuting subspaces of A_1 and simple left ideals in $M_2(k)$.*
- (2) *Each commuting subspace of A_1 annihilates a unique 1-dimensional subspace of $(M_{\tau'} \otimes k^2)_0$ and each 1-dimensional subspace of $(M_{\tau'} \otimes k^2)_0$ is annihilated by a unique commuting subspace.*
- (3) *If the fat point $\tau' + E[2]$ lies on a “line” L , then L is a commuting line.*
- (4) *The only fat point $p + E[2]$ that lies on a commuting line is $\tau' + E[2]$.*

Proof. (1) Since $E[4] = E \cap \{x_0x_1x_2x_3 = 0\}$ and $abc \neq 0$, f is a linear isomorphism by [Lemma 11.1\(2\)](#).

Let w_0, w_1, w_2, w_3 be the basis for A_1^* dual to the basis y_0, y_1, y_2, y_3 . By [Proposition 5.7](#), the commuting subspaces of A_1 are the planes

$$(abcw_0 + iaw_1) - t(ibw_2 - cw_3) = t(abcw_0 - iaw_1) - (ibw_2 + cw_3) = 0, \quad t \in \mathbb{P}^1.$$

The union of these subspaces is the quadric $\alpha\beta\gamma w_0^2 + \alpha w_1^2 + \beta w_2^2 + \gamma w_3^2 = 0$.

If $y = \sum \lambda_j y_j \in A_1$, then $\det f(y) = \alpha\beta\gamma\lambda_0^2 + \alpha\lambda_1^2 + \beta\lambda_2^2 + \gamma\lambda_3^2$. Hence f sends the quadric $\alpha\beta\gamma w_0^2 + \alpha w_1^2 + \beta w_2^2 + \gamma w_3^2 = 0$ in $\mathbb{P}(A_1)$ isomorphically to the quadric $\{\det = 0\}$ in $\mathbb{P}(M_2(k))$ and sends the ruling by commuting subspaces to one of the rulings on $\{\det = 0\}$. One of the rulings on $\{\det = 0\}$ is given by the simple left ideals in $M_2(k)$, the other by the simple right ideals.

The commuting subspace spanned by $iy_0 + bcy_1$ and $cy_2 + iby_3$ is sent by f to the linear span of

$$f(iy_0 + bcy_1) = \begin{pmatrix} 2iabc & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f(cy_2 + iby_3) = \begin{pmatrix} 0 & 0 \\ 2ibc & 0 \end{pmatrix}$$

which is a *left* ideal. Hence f sends commuting subspaces of A_1 to simple left ideals of $M_2(k)$.

(2) This follows from [Lemma 11.1\(1\)](#) because every simple left ideal in $M_2(k)$ annihilates a unique 1-dimensional subspace of k^2 and every 1-dimensional subspace of k^2 is annihilated by a unique simple left ideal.

(3) Let L be a line module for A for which there is a non-zero homomorphism $\phi : L \rightarrow M_{\tau'} \otimes k^2$. There is a 2-dimensional subspace $ky + ky' \subseteq A_1$ such that $L = A/Ay + Ay'$. Hence $ky + ky'$ annihilates a non-zero element $e \otimes v \in (M_{\tau'} \otimes k^2)_0$. By (2), $e \otimes v$ is also annihilated by a commuting subspace of A_1 . However, by [9, Prop. 8.6], $A_1(e \otimes v) = (M_{\tau'} \otimes k^2)_1$ which is 2-dimensional. It follows that $ky + ky'$ is the commuting subspace of A_1 that annihilates $e \otimes v$. Thus, L is a commuting line module.

(4) To show there are no other points in $E/E[2]$ that lie on a commuting line we must show that if $q \in E - (\tau' + E[2])$, then $\text{Hom}_{\text{Gr}(A)}(L, M_q \otimes k^2) = 0$. This follows from (2) and [Lemma 11.1\(3\)](#). ■

Theorem 11.3. *For $j \in \{0, 1, 2, 3\}$, $\tau' + \varepsilon_j + E[2]$ is the unique fat point in $\text{Proj}_{nc}(A)$ that lies on all the lines in $\text{Proj}_{nc}(A)$ that are parametrized by the conic C_j . There are no other incidence relations between conic lines and fat points in $E/E[2] = \text{Proj}_{nc}(\tilde{B})$.*

Proof. By [Proposition 11.2](#), the result is true for the commuting conic C_0 .

We use the quantum symmetry technique described in [Section 6](#) to transfer the result from C_0 to the other conics. By [Proposition 10.3](#), the action of the autoequivalence group $E[4]$ permutes the conics C_j or, more precisely, the line modules they parametrize, and preserves the variety $E/E[2]$

parametrizing fat \tilde{B} -points. By [Proposition 10.4](#), the four fat points $\tau' + \varepsilon_j + E[2]$ form an orbit under the action of the two-torsion subgroup $E[4]/E[2]$ of $E/E[2]$. ■

Corollary 11.4. *Let $(i, j) \in \{0, 1, 2, 3\} \times \{1, 2, 3\}$. The image of $E/\langle \xi_j \rangle$ in $\mathbb{G}(1, 3)$ meets C_i at 2 points. In total, 6 of the conic lines parametrized by C_i are elliptic lines, two for each $E/\langle \xi_j \rangle$.*

Proof. Let $p_i \in E$ be the point such that every line parametrized by C_i passes through the fat point module $M_{p_i} \otimes k^2$. By [Proposition 10.2](#), exactly two elliptic lines in each family $E/\langle \xi_j \rangle$ pass through that point.

If $i = 0$, [Proposition 11.2\(3\)](#) tells us those elliptic lines are, in fact, conic lines. Hence C_0 meets the image of $E/\langle \xi_j \rangle$ in $\mathbb{G}(1, 3)$ at exactly 2 points. By [Proposition 7.3](#), the action of $E[4]$ as auto-equivalence of $\text{Gr}(A)$ permutes the four conics and sends the set of line modules parametrized by $E/\langle \xi_j \rangle$ to itself. The result follows. ■

11.1.1. *Remark.* The 24 points in $\mathbb{G}(1, 3)$ that belong to

$$\bigcup_{\substack{0 \leq i \leq 3 \\ 1 \leq j \leq 3}} C_i \cap (E/\langle \xi_j \rangle)$$

are given in [Table 6](#). Since $E/\langle \xi_j \rangle$ parametrizes lines in $\mathbb{P}(S_1^*)$ of the form $\overline{p, p + \xi_j}$ we obtain, in this way, 24 distinguished secant lines in $\mathbb{P}(S_1^*)$.

12. EXACT SEQUENCES ARISING FROM INCIDENCES

In order to get a more complete picture of the incidence geometry of A we now identify the kernels of the surjections from elliptic and conic lines to points and fat points.

Throughout this section we write $\Gamma = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ where $\gamma_0 = 1$ and γ_j fixes x_0 and x_j . As before, we identify Γ with $E[2]$.

12.1. Ordinary points. Throughout [§12.1](#) we fix $j \in \{0, 1, 2, 3\}$ and an ordinary point module $P \in \text{Gr}(A)$ that corresponds to a point in \mathfrak{P}_j . Let $F = \tilde{V}(\tau + \xi_j)$. Thus F is a 1-critical S -module of multiplicity two and, by [Corollary 4.5](#), there is a Γ -equivariant structure on $F \otimes k^2$ such that $P \cong (F \otimes k^2)^\Gamma$.

If $p + q = \tau + \xi_j$, there is a non-zero homomorphism $M_{p,q} \rightarrow F$.

By [Theorem 9.2\(1\)](#), P is a quotient in $\text{QGr}(A)$ of exactly three elliptic lines, one in each of the three elliptic families. By [Theorem 8.1](#), P is also a quotient of exactly three conic lines, one in each of three conic families C_i , $i \neq j$.

The following result shows that the kernels of the resulting surjections from elliptic A -lines onto P are shifted conic lines and vice versa.

Recall $E[4]$ acts on the category $\text{Gr}(A)$ and the restriction of the action to $E[2] = \Gamma$ is induced by the action of Γ as automorphisms of A . For $\varepsilon \in E[4]$ we write ε^* for corresponding autoequivalence.

Throughout [§12.1](#) whenever we work with some general element $\xi \in E[2]$ we denote the other two non-zero elements of $E[2]$ by ξ' and ξ'' .

Proposition 12.1. *The kernels of the surjections from line modules onto the ordinary point modules are as follows.*

- (1) *If $P \in \mathfrak{P}_j$ is a quotient of a line module M in the $E/\langle \xi \rangle$ -family, there is an exact sequence*

$$0 \rightarrow L(-1) \rightarrow M \rightarrow P \rightarrow 0$$

in which L is the line module in the C_k -family that maps onto $\xi_k^ P$, where $\xi_k = \xi + \xi_j$.*

- (2) *If $P \in \mathfrak{P}_j$ is a quotient of a line module L in the C_i -family,⁵ there is an exact sequence*

$$0 \rightarrow M(-1) \rightarrow L \rightarrow P \rightarrow 0$$

⁵Recall that this implies $i \neq j$.

in which M is a line module in the $E/\langle \xi_i + \xi_j \rangle$ -family that corresponds to a Γ -equivariant structure on $(M_{p-2\tau, p+\xi-2\tau} \oplus M_{p+\xi'-2\tau, p+\xi''-2\tau}) \otimes k^2$ where $\xi = \xi_i + \xi_j$ and $2p = \tau + \xi_k$.

Proof. (1) The kernel of the surjection $M \rightarrow P$ is equal to $L(-1)$ for some line module L . There is a point $p \in E$ such that $2p + \xi = \tau + \xi_j$ and M corresponds to an equivariant structure on $(M_{p, p+\xi} \oplus M_{p+\xi', p+\xi''}) \otimes k^2$. By the proof of [Theorem 9.2](#), the surjection $M \rightarrow P$ comes from a Γ -equivariant homomorphism of S' -modules

$$(M_{p, p+\xi} \oplus M_{p+\xi', p+\xi''}) \otimes k^2 \rightarrow F \otimes k^2.$$

Its kernel K must be a Γ -equivariant S' -module and $K^\Gamma \cong L(-1)$.

Claim 1: L is a conic line.

Suppose to the contrary that L is an elliptic line module. Then $K \cong (M_{q, q+\omega} \oplus M_{q+\omega', q+\omega''}) \otimes k^2$ for some $q \in E$ where $\{\omega, \omega', \omega''\} = E[2] - \{o\}$. There would then be a non-zero morphism in $\text{Gr}(S)$ from $M_{q, q+\omega}(-1)$ or $M_{q+\omega', q+\omega''}(-1)$ to either $M_{p, p+\xi}$ or $M_{p+\xi', p+\xi''}$, and its cokernel would be a point module. This, however, contradicts the fact that when $u + v \notin E[2]$, which is the case for $\{u, v\} = \{p, p+\xi\}$ and $\{u, v\} = \{p+\xi', p+\xi''\}$, the only point modules that are quotients of $M_{u, v}$ are those arising from the exact sequences

$$0 \rightarrow M_{u+\tau, v-\tau}(-1) \rightarrow M_{u, v} \rightarrow M_u \rightarrow 0$$

and the analogous one for M_v . This proves Claim 1.

Claim 2: L surjects onto $(\xi + \xi_j)^*P$. By [Proposition 7.3\(1\)](#), $E[4]$ acts on $E/\langle \xi \rangle$ as translations by the image of $E[4]$ through $E \rightarrow E/\langle \xi \rangle$. Hence $\xi^*M \cong M$. Thus, applying ξ^* to $M \rightarrow P$ results in a surjection $M \rightarrow \xi^*P$. Since P is an ordinary point module, $\xi^*P \not\cong P$. The composition $L(-1) \rightarrow M \rightarrow \xi^*P$ is therefore non-zero. Hence there is a surjection $L \rightarrow (\xi^*P)(1)_{\geq 0} \cong \gamma_j^* \xi^*P$ (see the remark after [\(4.1\)](#)). Thus, L maps onto $(\xi + \xi_j)^*P$ as claimed.

Claim 3: L belongs to the conic family C_k , where $\xi_k = \xi + \xi_j$. Let C_k be the conic family to which L belongs. We will show that $\xi_k = \xi + \xi_j$.

Since P is in \mathfrak{P}_j , it is annihilated by $\Omega(\tau + \xi_j)$. Since the line module M is annihilated by a unique $\Omega(z)$, it must be annihilated by $\Omega(\tau + \xi_j)$. Therefore, if $\Omega = \Omega(z) \neq \Omega(\tau + \xi_j)$, $M/\Omega M$ is a \tilde{B} -module with Hilbert series $(1 - t^2)(1 - t)^{-2}$ so is isomorphic in $\text{QGr}(A)$ to $M_q \otimes k^2$ for some $q \in E$. As in the proof of Claim 2, the composition $L(-1) \rightarrow M \rightarrow M_q \otimes k^2$ leads to a surjection $L \rightarrow (M_q \otimes k^2)(1)_{\geq 0} \cong M_{q-\tau} \otimes k^2$. Since L is in C_k , [Theorem 11.3](#) tells us that $q - \tau \in \tau' + \varepsilon_k + E[2]$. By [Proposition 2.14](#), $2\tau' = -\tau$ so $2q = \tau + \xi_k$. By [Proposition 10.2](#) and the remarks before it, since $M_q \otimes k^2$ is a quotient of M , $q \in p + E[2]$. But $2p = \tau + \xi_j + \xi$ so $\tau + \xi_k = 2q = \tau + \xi + \xi_j$. The claim follows.

(2) Consider for a moment the proof of (1). The $\langle \xi \rangle$ -equivariant morphisms $M_{p, p+\xi} \otimes k^2 \rightarrow F \otimes k^2$ and $M_{p+\xi', p+\xi''} \otimes k^2 \rightarrow F \otimes k^2$ give rise to $\langle \xi \rangle$ -equivariant structures on the kernels which are, by [Proposition 2.19](#), $M_{p-2\tau, p+\xi-2\tau}(-2) \otimes k^2$ and $M_{p+\xi'-2\tau, p+\xi''-2\tau}(-2) \otimes k^2$ respectively. The action of ξ' then interchanges these two kernels, giving rise to a Γ -equivariant structure on their direct sum and hence to a shifted elliptic line module $M(-2)$, with M as in the statement of (2).

We have an embedding $M(-2) \subset L(-1)$ and so $L/M(-1)$ is a point module which must necessarily coincide with P since a conic line module admits a single surjection onto a point module, up to isomorphism ([Theorem 8.1](#)).

To see that every surjection $L \rightarrow P$ as in the statement of (2) fits into this framework simply observe that the group $E[4]$ of autoequivalences of $\text{Gr}(A)$ acts transitively on the set of isomorphism classes of such surjections. ■

12.2. Special points. Let P be a special point module. By [Theorem 9.2](#), if L is a conic line, then $\text{Hom}_{\text{Gr}(A)}(L, P) = 0$. In [§12.2](#) we describe the kernels of the surjections $M \rightarrow P$ when M is an elliptic line module.

Under the equivalence between $\mathrm{Gr}(A)$ and $\mathrm{Gr}(S')^\Gamma$, $\{P \mid P \in \mathfrak{P}_\infty\}$ corresponds to

$$\{(M_e \oplus M_e) \otimes k^2 \in \mathrm{Gr}(S')^\Gamma \mid M_e \text{ is a special point module for } S'\}.$$

Each M_e is annihilated by three of the four x_j 's so each special point module for A is annihilated by three of the y_j 's.

Proposition 12.2. *Let M_x be the line module corresponding to a point $x \in E/\langle \xi_i \rangle$. Let $P \in \mathrm{Gr}(A)$ be the special point module that is annihilated by $\{y_0, y_1, y_2, y_3\} - \{y_j\}$. If $f : M_x \rightarrow P$ is a surjection in $\mathrm{Gr}(A)$, then*

$$(12-1) \quad \ker(f : M_x \rightarrow P) \cong \begin{cases} M_{x-\tau}(-1) & \text{if } j \notin \{i, 0\}, \\ M_{x+\xi'_i-\tau}(-1) & \text{if } j \in \{i, 0\}. \end{cases}$$

Proof. Since i and j are fixed we drop the index i in ξ_i and write $E[2] = \{o, \xi = \xi_i, \xi', \xi''\}$, and we write M_e for the point module in $\mathrm{Gr}(S)$ that is annihilated by $\{x_0, x_1, x_2, x_3\} - \{x_j\}$ (see Section 9).

Let $p \in E$ be one of the two preimages of x .

The homomorphism $f : M_x \rightarrow P$ corresponds to a morphism

$$(12-2) \quad (M_{p,p+\xi} \oplus M_{p+\xi',p+\xi''}) \otimes k^2 \rightarrow (M_e \oplus M_e) \otimes k^2,$$

of Γ -equivariant S' -modules. Let K denote the kernel and M the domain of this map. Thus, $\ker(f) = K^\Gamma$ and $M_x = M^\Gamma$.

By [22, Thm. 5.7], the kernel of every surjective homomorphism $M_{p,p+\xi} \rightarrow M_e$ in $\mathrm{Gr}(S)$ is isomorphic to $M_{p-\tau,p+\xi-\tau}(-1)$. Thus, K is isomorphic to the S' -module $(M_{p-\tau,p+\xi-\tau} \oplus M_{p+\xi'-\tau,p+\xi''-\tau}) \otimes k^2$ equipped with one of its two Γ -equivariant structures. We will show that the two possibilities correspond to the two cases in (12-1). We will prove only the first statement, $j \notin \{0, i\}$; the other case is entirely analogous.

Since $M_{p,p+\xi}$ surjects onto M_e , the subspace of A_1 that annihilates $(M_{p,p+\xi})_0$ is contained in the linear span of $\{x_0, x_1, x_2, x_3\} - \{x_j\}$. By [9, Lemma 11.7], this subspace is spanned by an element of the form $\beta_0 x_0 + \beta_i x_i$ and x_k where $k \notin \{0, i, j\}$. Moreover, since $(M_{p+\xi',p+\xi''})_0$ is annihilated by $\beta_0 x_0 - \beta_i x_i$ and x_k , β_0 and β_i are both non-zero.

It follows that x_0 annihilates $(M_e)_0$ but not $(M_{p,p+\xi})_0$ or $(M_{p+\xi',p+\xi''})_0$. Hence $K_1 = x_0 M_0$.

Let $q + \langle \xi \rangle \in E/\langle \xi \rangle$. As in [9, §11.4], we define the line module corresponding to $q + \langle \xi \rangle$ as the Γ -invariants for the equivariant structure on $(M_{q,q+\xi} \oplus M_{q+\xi',q+\xi''}) \otimes k^2$ in which ξ_i fixes $(M_{q,q+\xi})_0 \otimes \binom{0}{1}$ and changes the sign of $(M_{q+\xi',q+\xi''})_0 \otimes \binom{0}{1}$. Since x_0 is fixed by Γ , $K^\Gamma \cong M_{x-\tau}(-1)$. ■

12.3. Fat point modules. As before, $M_p \otimes k^2$ is the fat point module for \tilde{B} corresponding to $p + E[2]$. If L is a line module and $f : L \rightarrow M_p \otimes k^2$ is non-zero we will show that $\ker(f) \cong L(-2)$.

Definition 12.3. The action of a group Γ on a category \mathcal{C} is *weakly trivial* if $\gamma^* c \cong c$ for all $\gamma \in \Gamma$ and all $c \in \mathcal{C}$. ◆

Lemma 12.4. *The action of Γ on $\mathrm{QGr}(\tilde{B})$ is weakly trivial.*

Proof. By [9, §8], the fat point module $M_p \otimes k^2$ corresponds to the skyscraper sheaf $\mathcal{O}_{p+E[2]}$ under the equivalence $\mathrm{QGr}(\tilde{B}) \equiv \mathrm{Qcoh}(E/E[2])$.

Let $\gamma \in \Gamma$. By [7, Theorem 5.4] the autoequivalence γ^* of $\mathrm{Qcoh}(E/E[2])$ is isomorphic to $\alpha^* \circ (\mathcal{L} \otimes -)$ for some automorphism $\alpha : E/E[2] \rightarrow E/E[2]$ and some line bundle \mathcal{L} . By Proposition 10.4, $\gamma^*(M_p \otimes k^2) \cong M_p \otimes k^2$; since $\mathcal{L} \otimes \mathcal{O}_{p+E[2]} \cong \mathcal{O}_{p+E[2]}$, α must be the identity.

However, γ^* also preserves the isomorphism class of $\tilde{B} \in \mathrm{QGr}(\tilde{B}) \equiv \mathrm{Qcoh}(E/E[2])$. Since the latter object is identified with the non-trivial self-extension \mathcal{V} of $\mathcal{O}_{E/E[2]}$ ([9, Prop. 8.4]), $\gamma^* \tilde{B}$ is a non-trivial self-extension of \mathcal{L} . This can only be isomorphic to \mathcal{V} when $\mathcal{L} \cong \mathcal{O}_{E/E[2]}$. ■

Proposition 12.5. *No line module for A is a \tilde{B} -module.*

Proof. By [Proposition 7.3](#), no points of the line scheme are fixed by Γ . Thus, if L is a line module $\gamma^*L \not\cong L$. Hence, as objects in $\text{QGr}(A)$, $\gamma^*L \not\cong L$. It therefore follows from [Lemma 12.4](#) that L is not a \tilde{B} -module. \blacksquare

Theorem 12.6. *If L is a line module, then the kernel of every non-zero morphism $L \rightarrow M_p \otimes k^2$ is isomorphic to $L(-2)$.*

Proof. It follows from [Proposition 12.5](#) that there is a non-zero degree-two central element Ω that does not annihilate L and hence acts faithfully on it. But Ω annihilates $M_p \otimes k^2$ so the image of $L \rightarrow M_p \otimes k^2$ must be isomorphic to $L/\Omega L$ by a Hilbert series comparison. In conclusion, the kernel of $L \rightarrow M_p \otimes k^2$ is $\Omega L \cong L(-2)$. \blacksquare

APPENDIX A. THE COMPONENTS OF THE LINE SCHEME

DEREK TOMLIN⁶

We use the method in [28] to determine the reduced line scheme, \mathbb{L}_{red} , for the algebras A defined in §3.1. For more detail about these methods the reader is referred to [8].

We use the coordinate functions y_0, y_1, y_2, y_3 on $\mathbb{P}(A_1^*)$ used earlier in this paper. We associate to the line in $\mathbb{P}(A_1^*)$ through different points (a_0, \dots, a_3) and (b_0, \dots, b_3) the point $(X_{01}, X_{02}, X_{03}, X_{12}, X_{13}, X_{23}) \in \mathbb{P}^5$ where $X_{ij} := a_i b_j - a_j b_i$. Thus X_{ij} is a 2×2 minor of

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

The Plücker polynomial $X_{01}X_{23} - X_{02}X_{13} + X_{03}X_{12}$ vanishes on this matrix. The Grassmannian of lines in $\mathbb{P}(A_1^*)$ is the zero locus in \mathbb{P}^5 of the Plücker polynomial.

Theorem A.1. [42] *The line variety, \mathbb{L}_{red} , of A is the union of four plane conics and three quartic elliptic curves:*

$$\Lambda_0: X_{01} + \alpha X_{23} = X_{02} - \beta X_{13} = X_{03} + \gamma X_{12} = X_{01}^2 + \alpha \gamma X_{12}^2 + \alpha \beta X_{13}^2 = 0.$$

$$\Lambda_1: X_{01} - \alpha X_{23} = X_{02} + X_{13} = X_{03} + X_{12} = X_{01}^2 - \alpha X_{12}^2 + \alpha X_{13}^2 = 0.$$

$$\Lambda_2: X_{01} + X_{23} = X_{02} + \beta X_{13} = X_{03} - X_{12} = X_{01}^2 - X_{12}^2 - \beta X_{13}^2 = 0.$$

$$\Lambda_3: X_{01} - X_{23} = X_{02} - X_{13} = X_{03} - \gamma X_{12} = X_{01}^2 + \gamma X_{12}^2 - X_{13}^2 = 0.$$

$$\begin{aligned} E_1: X_{01} = X_{23} = X_{02}X_{13} - X_{03}X_{12} \\ = (1 + \gamma)X_{02}^2 - (1 - \beta)X_{03}^2 - \gamma(1 - \beta)X_{12}^2 - \beta(1 + \gamma)X_{13}^2 = 0. \end{aligned}$$

$$\begin{aligned} E_2: X_{02} = X_{13} = X_{01}X_{23} + X_{03}X_{12} \\ = (1 - \gamma)X_{01}^2 - (1 + \alpha)X_{03}^2 + \gamma(1 + \alpha)X_{12}^2 + \alpha(1 - \gamma)X_{23}^2 = 0. \end{aligned}$$

$$\begin{aligned} E_3: X_{03} = X_{12} = X_{01}X_{23} - X_{02}X_{13} \\ = (1 + \beta)X_{01}^2 - (1 - \alpha)X_{02}^2 - \beta(1 - \alpha)X_{13}^2 - \alpha(1 + \beta)X_{23}^2 = 0. \end{aligned}$$

There is an isomorphism from $\mathbb{G}(1, 3)$ with coordinate functions X_{pq} to $\mathbb{G}(1, 3)$ with coordinate functions z_{rs} , which are used in §§5 and 6, given by $X_{pq} \mapsto z_{rs}$ whenever $\begin{pmatrix} 0 & 1 & 2 & 3 \\ p & q & r & s \end{pmatrix}$ is an even permutation. In the z -coordinates, the equations cutting out the four conics are

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$$\begin{aligned}
\Lambda_0: \quad z_{23} + \alpha z_{01} &= z_{13} - \beta z_{02} = z_{12} + \gamma z_{03} = \alpha z_{01}^2 + \beta z_{02}^2 + \gamma z_{03}^2 = 0; \\
\Lambda_1: \quad z_{23} - \alpha z_{01} &= z_{13} + z_{02} = z_{12} + z_{03} = \alpha z_{01}^2 + z_{02}^2 - z_{03}^2 = 0; \\
\Lambda_2: \quad z_{23} + z_{01} &= z_{13} + \beta z_{02} = z_{12} - z_{03} = -z_{01}^2 + \beta z_{02}^2 + z_{03}^2 = 0; \\
\Lambda_3: \quad z_{23} - z_{01} &= z_{13} - z_{02} = z_{12} - \gamma z_{03} = z_{01}^2 - z_{02}^2 + \gamma z_{03}^2 = 0.
\end{aligned}$$

These are the same as the equations for the conics in [Proposition 6.11](#), so $\Lambda_i = C_i$ for $i = 0, 1, 2, 3$. In terms of the z -coordinates, the three elliptic curves are

$$\begin{aligned}
E_1: \quad z_{23} &= z_{01} = z_{13}z_{02} - z_{12}z_{03} \\
&= (1 + \gamma)z_{13}^2 - (1 - \beta)z_{12}^2 - \gamma(1 - \beta)z_{03}^2 - \beta(1 + \gamma)z_{02}^2 = 0; \\
E_2: \quad z_{13} &= z_{02} = z_{23}z_{01} + z_{12}z_{03} \\
&= (1 - \gamma)z_{23}^2 - (1 + \alpha)z_{12}^2 + \gamma(1 + \alpha)z_{03}^2 + \alpha(1 - \gamma)z_{01}^2 = 0; \\
E_3: \quad z_{12} &= z_{03} = z_{23}z_{01} - z_{13}z_{02} \\
&= (1 + \beta)z_{23}^2 - (1 - \alpha)z_{13}^2 - \beta(1 - \alpha)z_{02}^2 - \alpha(1 + \beta)z_{01}^2 = 0.
\end{aligned}$$

By [Proposition 7.2](#), $E_j \cong E/\langle \xi_j \rangle$.

Corollary A.2. [42] *Each C_i , $i = 0, 1, 2, 3$, intersects each E_j , $j = 1, 2, 3$, at two points. With respect to the ordered coordinate functions $(z_{01}, z_{02}, z_{03}, z_{12}, z_{13}, z_{23})$, the two points in $C_i \cap E_j$ are those in row C_i and column E_j of Table 6.*

	E_1	E_2	E_3
C_0	$(0, c, \pm ib, \mp ib\gamma, \beta c, 0)$	$(c, 0, \pm ia, \mp ia\gamma, 0, -\alpha c)$	$(b, \pm ia, 0, 0 \pm ia\beta, -\alpha b)$
C_1	$(0, 1, \pm 1, \mp 1, -1, 0)$	$(1, 0, \pm a, \mp a, 0, \alpha)$	$(1, \pm ia, 0, 0, \mp ia, \alpha)$
C_2	$(0, 1, \pm ib, \pm ib, -\beta, 0)$	$(1, 0, \pm 1, \pm 1, 0, -1)$	$(b, \pm 1, 0, 0, \mp \beta, -b)$
C_3	$(0, c, \pm 1, \pm \gamma, c, 0)$	$(c, 0, \pm i, \pm i\gamma, 0, c)$	$(1, \pm 1, 0, 0, \pm 1, 1)$

TABLE 6. Intersection points $C_i \cap E_j$.

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